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EDITED BY

T. A. A. BROADBENT, M.A.

Royal Naval College, Greenwich, London, S.E. 10

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AN ENGLISH SCHOOLMASTER LOOKS AT AMERICAN MATHEMATICS TEACHING*

By W. S. Brace.

During the school year 1950–51 I had the privilege of being one of the British teachers visiting the United States of America under the official exchange scheme. In addition I had the great good fortune to be assigned to Denver, Colorado, which must have as beautiful a situation as any city of its size in the world. Denver has a population of about half a million, and its school system is reputed to be one of the best " middle of the road " systems of the United States. I believe, therefore, that I had the pleasure of observing and sharing in American education at its best.

The first point which I must make—and I cannot make it too forcefully—is that American education, like everything else in America, is diversified in the extreme. I cannot say with any certainty which of the things I saw are really typical of American education and which are peculiar to Denver or to the State of Colorado. I visited a number of schools—about fifty—in Denver and in different parts of the State of Colorado, and talked with teachers from many parts of the United States, but it was by no means easy for me to separate variations of detail from variations in the pattern of education. Here I shall simply give my impressions of what I saw and learnt in America and make appropriate comparisons with our own schools to point similarities and differences.

For any reasonable appreciation of American methods some knowledge of the American educational system, which is very different from our own, is essential. There is a Federal Office of Education in Washington, D.C., but it is one of the minor government agencies, and its functions are chiefly statistical and advisory. Education is in the hands of the individual States ; in educational matters the States are supreme. Since the States vary widely in population, industries and wealth, there is a wide variation in their education laws—variation in leaving age, financial arrangements, teachers qualifications, and so on. For example, the minimum leaving age varies from fourteen to eighteen ; in Colorado the law was " when the age of fifteen is reached, or when the eighth grade has been completed ". Most States maintain Offices

* A paper read to the Sheffield Branch of the Mathematical Association, May 28, 1952.

of Education, but these have, it seemed to me, less influence on education in their States than has the Ministry of Education in Britain.

The units of educational administration are the School Boards, comparable to the school boards which we had in the nineteenth century, and catering for limited areas, frequently with small scattered rural populations. By numbers the commonest American schools are still all age one or two teacher schools. On the other hand, the city of Denver has only one School Board, and schools numbering their pupils in hundreds. The members of the school boards are directly elected for various terms, and their chief executive is termed a superintendent. The superintendent and his attendant administrators interfere a good deal more in the day to day life of the schools than do Chief Education Officers and their assistants in England. For example, it is quite common for courses of study and text-books to be prescribed by the administration.

The schools are generally comprehensive and coeducational. By that I mean that all pupils of a given age in a given area normally attend the same school. This means that where numbers allow—that is, in the towns—schools are large. A common way of dividing the schools is : grade schools for pupils from six to twelve ; junior high schools, for pupils from thirteen to fifteen ; senior high schools for pupils from sixteen to eighteen ; but other modes of division are also used. Grade schools may have up to three or four hundred pupils ; junior high schools up to one thousand pupils ; and I heard of senior high schools with as many as ten thousand pupils. I myself taught in a senior high school with 2,500 pupils. In the grade schools the pupils have the same teacher for all, or almost all subjects, as in our primary schools ; in the junior high schools there is some subject teaching, as in our secondary modern schools ; in the senior high schools all teaching is subject teaching, as in our grammar schools. Large schools are needed to provide an adequate range of courses to suit all intellectual levels and to attract the interests of the pupils. This is considered of great importance, and schools are partly judged in the public eye by their ability to attract pupils after the age of compulsory attendance.

American educational objectives are stated more explicitly, more precisely, and more frequently than are our aims. One popular statement of the goals of American education is headed " Imperative Needs of Youth ", and these are listed as

1. All youth need to develop saleable skills and those understandings and attitudes that make the worker an intelligent and productive participant in economic life. To this end, most youth need supervised work experience as well as education in the skills and knowledge of their occupations.

2. All youth need to develop and maintain good health and physical fitness.

3. All youth need to understand the rights and duties of the citizen of a democratic society, and to be diligent and competent in the performance of their obligations as members of the community and citizens of the state and nation.

4. All youth need to understand the significance of the family for the individual and society and the conditions conducive to successful family life.

5. All youth need to know how to purchase and use goods and service intelligently, understanding both the values received by the consumer and the economic consequences of their acts.

6. All youth need to understand the methods of science, the influence of science on human life, and the main scientific facts concerning the nature of the world and of man.

7. All youth need opportunities to develop their capacities to appreciate beauty in literature, art, music, and nature.

8. All youth need to be able to use their leisure time well and to budget it

wisely, balancing activities that yield satisfactions to the individual with those that are socially useful.

9. All youth need to develop respect for other persons, to grow in their insight into ethical values and principles, and to be able to live and work co-operatively with others.

10. All youth need to grow in their ability to think rationally, to express their thoughts clearly, and to read and listen with understanding.

(Quoted from *Planning for American Youth*, published for the National Association of Secondary School Principals.)

It is clear that the goal of satisfying these needs provides a clear and intelligible philosophy of education, though there is obviously much room for variation in emphasis. Further—as in all educational projects—what is planned is not necessarily what is executed.

Clearly mathematics as far as and including elementary arithmetic, knowledge of simple formulae, use of simple graphs and a little stage A geometry is needed to meet several of these needs. Consequently all pupils follow courses which cover these fundamentals. Beyond these basic skills there is an elaborate system of options, satisfying the needs of pupils with definite professional or vocational aims, and one of the most interesting and suggestive things about American education is the way in which these options are organised, and the way in which pupils are guided to exercise a wise choice among the many courses offered. In no case did I find the choices were made in the haphazard way which many English writers on American education suggest. At a recent education conference in this country I heard it said that " If a boy doesn't like Algebra, he drops it and takes up something else, such as learning to drive." In Denver, at least, that is not so, for most Senior High School pupils have driving lessons in school time as part of their General Education programme. Remember, in America almost everyone has a car, and correct handling of a car is a social responsibility.

The aim of each senior high school pupil is to gain a graduation diploma—what we should term an internal leaving certificate—and the requirements to be satisfied for such a diploma are carefully prescribed. For example, in Denver, during three years in senior high school, 15 units must be earned—a unit is a year's work, at the rate of one period daily—which must include 2 units of English, $1\frac{1}{2}$ units of Social Science, 1 unit of Mathematics, 1 unit of Physical Education, $\frac{1}{2}$ unit of Guidance, $\frac{1}{2}$ unit of Health, making a total of $6\frac{1}{2}$ units, and leaving $8\frac{1}{2}$ units at the pupil's disposal. The pupils will normally have two major fields of interest, in each of which he will do 3 units of work. For a unit to count the work must be done to the satisfaction of the teacher, and it is quite common for a teacher to refuse credit for a term's work which he considers unsatisfactory. Assessment is based on classwork, homework, performance in attainment tests and general co-operation.

Clearly the whole system depends on guiding pupils into appropriate courses. During his three years in senior high school each pupil has a member of staff for counsellor, with whom he has conferences at regular intervals. The counsellor is the teacher who conducts his guidance class, and frequently also conducts his social science classes. All teachers are encouraged to share in this work, and training in guidance is part of a teacher's professional preparation. Course selection is based on a consideration of the pupil's abilities and interests, his ambitions, his parents' wishes, graduation requirements, etc. ; the decisions are mutual decisions, not orders from above. Changes of course are not undertaken lightly, but do occur. The most obvious effect of this method of planning is that the majority of pupils have a coherent conception of the purpose of their school studies, and consequently require less urging to do the necessary modicum of distasteful toil. In fact one of my strongest impressions

on resuming teaching in this country was how aimless many of our fifth form pupils are compared with their American counterparts.

After this survey of the American educational system as I found it, perhaps I may turn to more specifically mathematical topics. Again the starting point must be course arrangements in a senior high school. Courses offered are of three types. First there are classes in remedial mathematics. These classes are organised under a variety of names, but all have as their objective the removal of defects in basic mathematical competence or understanding. Defects of this sort are, I believe, not unknown in English grammar schools! Recently I had a bitter argument with a fifth form pupil on the question whether the solution of the equation $x/3=6$ involved multiplication or division by 3! There appear to be two possible ways of dealing with these basic defects: one is to assume that previous teachers were negligent or incompetent, and did not give the pupil sufficient practice; the other is to conclude that the pupil was presented with certain material when he had not sufficient maturity for that material, as presented, to have meaning for him. The first is, of course, the classical attitude, and leads to drill and yet more drill; the second is the attitude of modern educational theorists. In America I was very much impressed by the excellent results being obtained in these remedial classes, with the equipment available to diagnose deficiencies and the care taken to identify the exact skills lacking, and with the trouble taken to remedy defects identified, but in spite of the enormous amount of research published on psychologically effective approaches and means of making mathematics meaningful, I always felt that the classical method was the one in use, and that the results of recent research were given little serious consideration.

The second type of class which is common is the general mathematics class. Ostensibly the philosophy of these classes is the philosophy behind the recommendations of the Jeffrey Report; that is, that mathematics should be taught as a whole, integrating the different branches into a comprehensive subject, and that the subject should be made relevant to modern life and intelligible to the pupils. There is the same attempt to prune away dead wood and strip off formal manipulation and mechanical routines. In practice these classes often turn out to be arithmetic with a very slight admixture of algebra —substitution in easy formulae and a few simple graphs, with the intention of introducing the idea of functionality—and a little geometry—some mensuration. The claim is that these classes are for all pupils, but in practice it is usually the weaker pupils who find their way into them. Again an excellent job is being done within the limitations accepted, and the teachers strive manfully to overcome the restrictions of the situation in which they find themselves. Nevertheless, these classes in practice seem to me to be rather different from those envisaged by their proponents.

The third type of class is the traditional formal mathematics class which we know so well. But the American practice is to teach one branch of mathematics at a time. First a year or a year and a half of a two years algebra course; then a year of plane geometry, followed by half a year of solid geometry; the remainder of the algebra course; then half a year of trigonometry; finally half a year devoted to analysis. In certain cases two of the later courses are taken concurrently, and there are proficiency bars at various stages. The classes move on solid traditional lines, and concentrate on giving a thorough training in the basic techniques of the various branches of mathematics. In the algebra classes particularly the sub-division of subject matter is not quite so complete as the titles of the classes seem to indicate, and some geometrical work is often included. Formal manipulation, however, is given more emphasis and is carried to higher levels of complexity than we are accus-

tomed to today. Standards are reasonable having regard to the age at which the studies are started and the time spent at them. If a subject is being studied it is studied for one period each day—timetables are designed on a daily basis instead of the weekly basis used in this country. This time allowance is exactly what we have up to the fifth forms, but is considerably less than we have in the sixth forms. The final level reached by the best pupils was rather higher than the subsidiary subject standard of the old Higher School Certificate. In a school of 2,500 about ten pupils each year reached this level—say one percent of the age group. We might compare this with an equal number of pupils reaching advanced G.C.E. level in a three stream grammar school. The result seemed to me to be quite reasonable, and compared well with what we find in the average English grammar school. Of course many English grammar schools reach a much higher standard quantitatively, but their pupils are usually subject to some form of selection beyond the usual special place examination.

This raises the whole thorny question of comparative qualitative standards in Britain and America. I regret to say that in my opinion the question bristles with so many difficulties that anyone who tries to be fair to both sides is almost certain to incur the displeasure of both. Any simple answer is, I believe, utterly misleading. Is a school to be criticised for not achieving what it is not trying to do? Are English schools to be condemned as being of a low standard because they make very little direct provision for civic training, or for instrumental music? Are American schools to be condemned for making places for things like these in their curricula? Your answer must depend on your philosophy of education, which in turn will depend on your philosophy of life.

I have implied that American pupils will normally start Algebra at the age of fifteen or sixteen, and start geometry at sixteen or seventeen. This seems late to us. On the other hand pupils with IQ's in the 90's will do this—of course they will be among the last to start, perhaps not until they are eighteen. The abler pupils make very rapid progress when they do start, and after eighteen months of algebra will reach G.C.E. ordinary level. In the same way, after a year's geometry they will reach G.C.E. ordinary level. The best pupils will reach this standard at about 16½ or 17, which is not far behind our best pupils, and they will have spent a good deal of time at studies which our best pupils have neglected. Thereafter they proceed along less specialised lines than most of our sixth form pupils, so naturally they do not reach the same level. But is the result any less desirable educationally? The battle of over-specialisation rages perpetually in our educational journals. The best answer I can give is that scholarship level work, or even second year sixth form work in mathematics is unknown in American high schools, but I am not satisfied that the result is undesirable educationally anywhere, and I am certain that it is appropriate in the American situation. It is essential to remember that what is done in a school is a reflection of the philosophy of education which prompts the school. Merely because schools are for young people, use blackboards and chalk, and so on, we must not assume that they are prompted by the same philosophies whether they are in England, Wales, Scotland, France or the United States. They are not, and if they are not, qualitative comparisons become impossible. I prefer an orange to a grapefruit—but even if you agree with me, does that prove that my wife, who doesn't, is wrong?

I have indicated roughly the extent of the various courses offered in the high schools. Perhaps a little more detail of the content of the courses would be of interest. In arithmetic there is a good deal of concern with "consumer arithmetic", by which is meant the elementary arithmetic of buying and selling, meter reading, insurance, banking, instalment buying and so on.

There is also some study of the arithmetic of taxation, both from the national and from the individual point of view. This part of the work is considerably more extensive than anything comparable found in our schools. Under arithmetic is also included elementary work with formulae and the mensuration of common areas and solids, including the circle, cylinder and prism, and some practical geometry and trigonometry. This course normally finishes when the pupil is 14 or 15, but may continue until the pupil leaves school.

The algebra course is a two year course, but most pupils take only the first year and a half. This year and a half includes manipulation of literal expressions, directed numbers, linear simple and simultaneous equations, quadratic equations in one unknown, exponents and radicals, functions and graphs, ratio and proportion—in fact very much the algebra of the old type G.C.E. ordinary papers. Complexity of manipulation, however, is considerably greater than we have practised for many years. The last half year, usually taken after an interval spent studying formal geometry, includes irrational equations, imaginary and complex numbers, the binomial theorem, permutations and combinations, progressions, and the theory of equations—a formidable list indeed for half a year's work. In practice some selection of material occurs, and the emphasis appears to be on elementary manipulation and application, rather than on theoretical development.

Geometry courses, again, match our G.C.E. ordinary courses in content, with perhaps a little more emphasis on the postulates and axioms and on inequalities and a little less emphasis on circles and similar figures than we are accustomed to. Solid geometry is first developed logically from appropriate postulates and axioms, and then a good deal of work is done in calculating lengths, angles, areas and volumes, especially in connection with prisms, cylinders, cones and spheres.

The trigonometry courses include solution of right and oblique triangles, functions of angles of any size, trigonometrical equations and identities. The field is about that of the N.U.J.M.B. A22 Mathematics, with rather more emphasis on numerical calculations.

The course in Analysis is very wide, and a selection is made from the following range of topics: functions and graphs, including simple discontinuities; differentiation and integration (as far as inverse trigonometric functions and exponential and logarithmic functions, but excluding integration other than by very simple substitutions; rectangular coordinates (very elementary work with central conics, referred to their centres, and the parabola); polar coordinates; solution of equations; definite integrals. The ground covered is very much less than that of our G.C.E. Advanced Mathematics.

I have not yet referred to the most interesting mathematics course I met during my American visit. When I arrived in Denver I was asked to take an "A" course in Geometry, and in all innocence I said that I would. Since this was to be the pupils' first contact with formal geometry I thought that I could manage it without undue strain. Judge my horror when I was presented with a text-book with the title "Clear Thinking", and found that "Geometry A" really meant a course in elementary applied logic, using geometric postulates, axioms and theorems as a convenient and easy field, but ranging far and wide in subject matter!

No doubt you all remember the efforts of the Association for the Reform of the Teaching of Geometry, the parent of our Association, and also the first report on "The Teaching of Geometry" published by our Association. To overcome the pertinent objection that, in view of the discoveries of experimental psychology, formal geometry could be considered of very little use to the majority, for mind training or anything else, unless its data and their

relation to the external world were well understood, the suggestion was made that the teaching of geometry should fall into three stages: Stage A, Experimental; Stage B, Deductive; Stage C, Systematising. We have tried to make formal geometry grow out of an exploration of space, so that even if our pupils do not train their minds, they will at least learn some practical geometry. My " Geometry A " course tackled the same problem from a different point of view. Some transfer of training can take place if teaching is directed to that end—that is if care is taken to emphasise, in this case, the reasoning processes used, and if examples of the application of these processes in a wide variety of fields is deliberately sought. Remember that this geometry course is for pupils of seventeen years, who have already reached G.C.E. ordinary level in algebra, are of fair intelligence, and have voluntarily agreed to take this particular course. I think that you must agree that for general education and culture this course is quite as sound as the one with which we are more familiar. Later I was told that until a teacher has spent at least three years at this course, building up a suitable body of examples, it is really hard work to teach. I certainly found that it required a lot more thought and preparation than any other course at any level that I have taken. I will also say that I got more satisfaction from this course in the end than from any other course, except possibly my real love—analysis with good scholarship pupils. The products of the course were listed by students as being: making and recognising generalisations; avoiding over-generalisations; understanding the need for generalisations; applying generalisations; use of indirect reasoning; recognising faulty forms of thinking, e.g. inadequate data, faulty analogies, misusing converses and inverses, thinking in circles, rationalising, word reasons, mistaking sequences for causes, begging the question; developing the habit of thinking before speaking; saying what is meant; noting double meanings; use of definitions; following directions precisely or with judgement; recognising implied assumptions; facing facts; etc. An imposing list! How many of our sixth form pupils get that, or even a small part of it from their study of mathematics? I suppose I should add that American teachers are at least as conservative as English teachers, and that I was asked to take the course partly because it was less popular than a more traditional alternative course! However, I repeat, I enjoyed the class, and believe that, in the field of teaching, it was one of my most valuable experiences.

On the whole I found that American teaching methods and courses tended to be more formal than our own. Teacher-pupil relations were rather freer than ours, but not freer than are found with individual teachers in most English schools. All the staff, all the pupils I taught and any others that I came into contact with greeted me every day when we met for the first time—true the greeting was the standard American " Hi! ", but I missed it very much when it was replaced by the traditional English dour silence. Pupil-teacher relationships were friendly, but there was an innate attitude of respect. Subject matter, on the other hand, was very formal. Lip service was paid to applications of mathematical methods, but the applications were few and despised or feared. The terror of any algebra class was the story problem; the textbooks I used had very few problems, and at the higher levels of study the lack made itself felt. Several pupils in my analysis class told me that their difficulties were largely due to lack of experience in and consequent fear of work of the problem type.

The same formality extended to " Geometry B ", the traditional geometry course, which was rather more formal than our Stage B, and began with a list of axioms, postulates, etc. to be learnt. I also found it rather difficult to discuss congruence with people who had little idea how to measure an angle,

and who felt aggrieved if they were asked to verify the results of reasoning by drawing and measurement. Faith in reasoning is good, but a realisation of the frailty of human reasoning is scientific! The solid geometry text I used had a number of very beautiful pictures of aeroplanes, bridges, and so on, with hints that solid geometry was needed in their construction, but I searched the book in vain for any clue indicating just what was used. I feel that our best text-books might not have the pictures, but that they would certainly have some examples of applications. It is only fair to add that, although none of the books I used was old, much more satisfactory texts are now appearing.

Teaching techniques seemed to me to be very much the same as in England. I met one or two ideas which were new to me, but which I felt I might easily have met here. One was the institution of a class president, vice-president and secretary, fulfilling many of the functions of our form-captains or monitors, but also accepting responsibility for checking attendances, calling the class to order, organising recitations, and so on. The matter of recitations I found interesting. For homework a set of examples would be set, and preparation would often be tested by calling on pupils to do selected examples on the board, and to defend their working against criticism by other pupils. The selection of pupils for this purpose, and the conduct of the cross-examination was in the hands of the class president. Of course the teacher added his criticism where necessary, but the pupils usually spotted all the points which called for criticism. Of course it was no novelty to see pupils working at the board, but to use it as a systematic method of checking homework in geometry or analysis was novel. I found the method effective. Blackboards were large, and a common practice was to get four or five pupils to prepare their figures and particular enunciations simultaneously, and then go through their proofs in order.

Generally I think that the standard of teaching in America is of the same order as in Britain, but the tendency for good teachers and highly qualified teachers to leave the schools is rather greater than it is here. Elevation to administrative heavens is the ambition of the good teacher, and a fair number make the grade. Nevertheless I met many skilful and capable teachers actually teaching. A further important point is that in many areas all teachers at all levels must be graduates. While admitting that a first degree in an American university is not quite the same thing as a first degree here, yet a university degree of the more generalised type popular there certainly has great value in equipping a teacher.

The conditions under which teachers have to work vary very widely—the best no better than our best, the worst no better than our worst, the average, little different from our average. I think schools are less well equipped for mathematics teaching than ours, perhaps because the teaching is more formal. Visual aids are available fairly freely, and I experimented with a number of film-strips borrowed from a common library, but I found, as with English strips, that they seldom did much that I could not do with equal facility on the blackboard, and with less waste of time and general disturbance to the class.

A point of some interest is the way in which some sort of uniformity between schools is achieved in the absence of anything resembling our General Certificate of Education examinations, for a broad uniformity certainly exists. There are two influences making for uniformity. The first is the use of standardised attainment tests. These tests exist in most subjects and at many levels—well over one hundred such tests were in use in Denver—and it is a common practice for a teacher to subject his pupils to one of these tests at the end of each half-year. He does this partly in self-defence, so that if there is any criticism of his teaching he can point to test results, and partly for his own

satisfaction, so that he may have a check on his own work. I must admit I found the tests quite helpful, as providing a simple measuring rod to determine the progress of my pupils. While I do not think that any teacher teaches for these tests as we teach for G.C.E., yet their existence in the background is a kind of framework for the courses.

A second influence for uniformity is provided by the universities. High school diplomas are accepted by the universities as evidence of fitness for entrance provided the high school fulfills certain conditions. These conditions include accommodation and qualifications of the staff. The universities also demand certain units of different subjects according to the particular school it is desired to enter, and for these units a certain minimum range of subject matter is expected. Naturally the text-books published are in line with these requirements, and so in spite of the absence of G.C.E. there is very nearly as much uniformity between schools in America as there is in this country.

To summarise my impressions—mathematics teaching in the United States is much more like our own than I had expected. Published research is not widely used, and, again contrary to expectations, teaching is rather more formal than in this country. On the other hand, course arrangements are much more flexible, and more students pursue their studies of mathematics to a reasonable level than happens here. Also, as a consequence of arrangements made to make the best use of the varied courses and of the close atmosphere of co-operation between teachers and pupils, studies are generally pursued with a good deal of vitality. However, under existing conditions, the best pupils do not proceed as far as our best pupils with their studies. In some ways the American system has the advantage, in some ways ours. I have been happy in both.

W. S. B.

GLEANINGS FAR AND NEAR

1812. When Found Make a Note of.
Doubts are cast by modern mathematicians on the universal validity of the conclusions reached by Euclid in his propositions, but at least he knew how to handle an argument, and always wound up with " This conclusion should be tested by practical experiment ".—Robert Graves and Alan Hodge, *The Reader over your Shoulder*, (Cape, 1947) p. 104. [Per Mr. J. C. W. De la Bere.]

1813. The Moscow Radio announced that five million Russians filed past Josef Stalin's bier in ·72 hours. That means, according to the calculations of Frank Baker, an accountant, that the mourners, two abreast, three and one-third feet apart, ran past the bier at 22 m.p.h. 22 m.p.h. is 9·3 seconds a hundred yards, which is the world's record for the 100 yard dash, heretofore recorded only by America's Mel Patton.—*Reader's Digest*, July 1953, p. 114. [Per Miss Dorothy Cox, Form V, Girls' High School, Eastbourne.]

1814. The first tool needed by any analysis is an appropriate language ; a language capable of describing the precise outlines of the facts, while preserving the necessary flexibility to adapt itself to further discoveries and, above all, a language which is neither vacillating nor ambiguous.—Marc Bloch, *The Historian's Craft* (English translation).

INCYCLIC-CIRCUMCYCLIC QUADRILATERALS

By E. H. Lockwood.

1. Let α, β, γ, δ be parameters of four points A, B, C, D on the in-circle $x^2+y^2=a^2$, the coordinates of A being $(a\cos\alpha,\ a\sin\alpha)$ and so on. (Fig. 1).

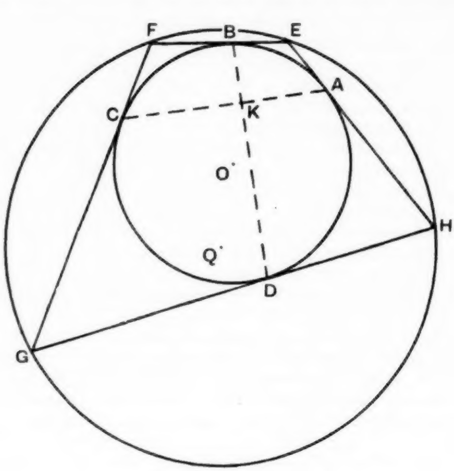


Fig. 1.

The tangent at A is $x\cos\alpha+y\sin\alpha-a=0$. The equation

$$(x\cos\alpha+y\sin\alpha-a)(x\cos\gamma+y\sin\gamma-a)$$
$$+(x\cos\beta+y\sin\beta-a)(x\cos\delta+y\sin\delta-a)=0 \quad\text{.........(1)}$$

represents a conic through the four intersections E, F, G, H of the tangents at A, B, C, D, and this conic will be a circle if

$$\cos(\alpha+\gamma)=-\cos(\beta+\delta) \quad\text{and}\quad \sin(\alpha+\gamma)=-\sin(\beta+\delta).$$

Both conditions are satisfied if $(\alpha+\gamma)\sim(\beta+\delta)=180^\circ$, ...(2)

i.e. if AC is perpendicular to BD.

Thus a circle (1) can be obtained for any three points A, B, C by constructing D so that BD is perpendicular to AC.

2. The circle (1) may be written

$$t(x^2+y^2)-2apx-2aqy+2a^2=0,$$

where

$$\left.\begin{aligned} t&=\cos\alpha\cos\gamma+\cos\beta\cos\delta,\\ &=\sin\alpha\sin\gamma+\sin\beta\sin\delta;\\ 2p&=\cos\alpha+\cos\beta+\cos\gamma+\cos\delta,\\ 2q&=\sin\alpha+\sin\beta+\sin\gamma+\sin\delta. \end{aligned}\right\} \quad\text{....................(3)}$$

From these, $4p^2+4q^2=4+2\Sigma\cos(\alpha-\beta)$;

but $\cos(\alpha-\beta)=-\cos(\gamma-\delta)$

and $\cos(\alpha-\delta)=-\cos(\beta-\gamma)$, by equation (2).

Thus $4p^2+4q^2=4+4t$; and $p^2+q^2=1+t$. ....(4)

This shows that, if p and q are constant, so also will be t, and the circumcircle (1) will remain fixed. But p, q define the centroid of the four points A, B, C, D, its coordinates being $(\frac{1}{2}ap, \frac{1}{2}aq)$, and this in turn fixes the intersection K of AC and BD, for the centroid is the mid-point of OK. It follows that, for any new position A' of A, we can construct B', C', D' by joining $A'K$, producing to C', and drawing $B'KD'$ at right angles to it. Tangents at these points will form a quadrilateral whose vertices lie on the circle (1). Thus if two circles have one incyclic-circumcyclic quadrilateral, they will have an infinite number of them.

3. The centre Q of the circumcircle has coordinates $(ap/t, aq/t)$, and O, K, Q are therefore collinear. Let $OQ = r$ and let the radius of the circumcircle be R.

Then $$r^2 = (ap/t)^2 + (aq/t)^2 = a^2(1+t)/t^2,$$

and $$R^2 = (ap/t)^2 + (aq/t)^2 - 2a^2/t = a^2(1-t)/t^2. \qquad \text{.................(5)}$$

By eliminating t, $$2a^2(r^2 + R^2) = (r^2 - R^2)^2,$$

whence $$a = \pm \frac{r^2 - R^2}{\sqrt{\{2(r^2 + R^2)\}}}, \quad \text{and} \quad R^2 = r^2 + a^2 \pm a\sqrt{(4r^2 + a^2)}. \qquad \text{.........(6)}$$

Given the radius a of the in-circle, any value of r may be chosen, for equation (6) will always give a value of R (two values if $r^2 > 2a^2$); so we may choose any point as centre of the circumcircle. Moreover r and R are interchangeable, t changing sign at the same time. t is negative when the circumcircle (centre Q in Fig. 2) encloses the incircle, i.e. when $R > r + a$; and is positive when it

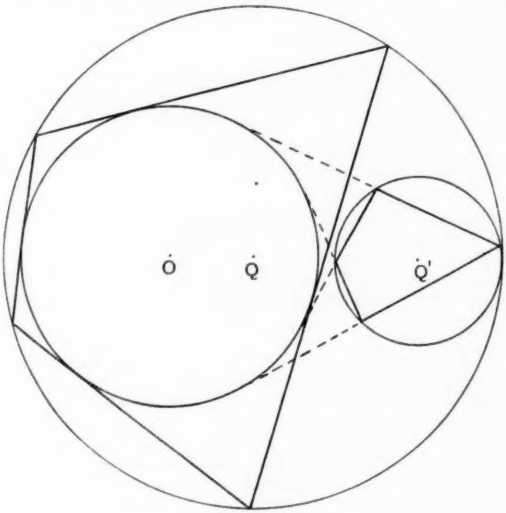


Fig. 2.

does not, i.e. when $R < r - a$. Fig. 2 shows the two circumcircles, centres Q, Q', drawn for a given incircle, centre O, the values of t being equal and opposite. This figure shows the special case in which O, Q, Q' are collinear; the two circles touch each other internally at a point distant $(r + R)$ from O. If Q, Q' were on opposite sides of O, they would touch externally, at a point distant $(r \sim R)$ from O. In all other positions they would intersect each other.

4. Given the two (unequal) circumcircles, touching or intersecting, it is possible to construct the incircle, as follows :

Let the given circles, with centres A, B, radii R, r, intersect at P, Q (Fig. 3).

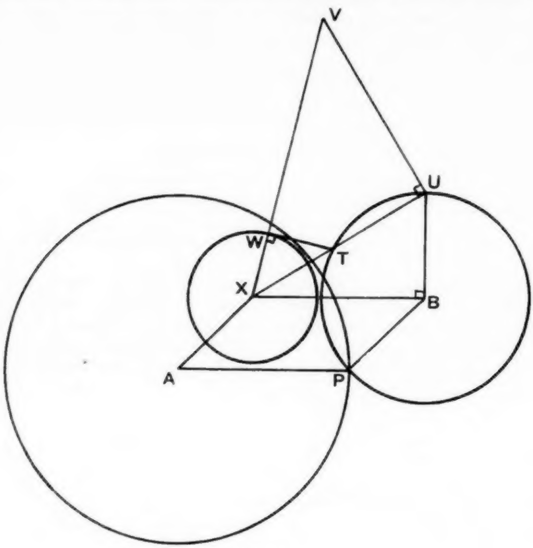


FIG. 3.

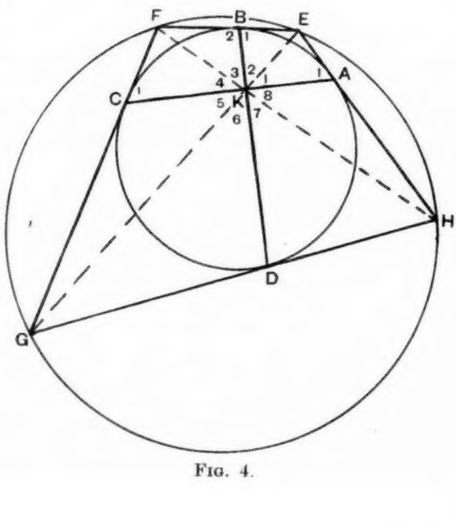


FIG. 4.

Complete the parallelogram *APBX* (or *AQBX'*). Draw a radius *BU* at right angles to *XB* and let *XU* meet again at *T* the circle whose centre is *B*. Construct $\triangle XUV$ with a right angle at *U* and *UV* equal to *XU*. Draw *TW* at right angles to *XV*, meeting it at *W*. Then a circle whose centre is *X*, radius *XW*, satisfies the conditions ; for its distances from *A* and *B* are *r*, *R* respectively, and its radius is equal to

$$XT \,.\, XU/XV = (R^2 \sim r^2)/\sqrt{\{2(R^2+r^2)\}}$$

5. It may also be noted that, if *EFGH* is one of a set of inscribed-circumscribed quadrilaterals, touching the incircle at *A*, *B*, *C*, *D* (Fig. 4), then its diagonals are concurrent at *K* with *AC* and *BD*. For in triangles *KEA*, *KEB* $EA = EB$ and *EK* is common ; hence

$$\frac{\sin K_1}{\sin K_2} = \frac{\sin A_1}{\sin B_1} = \frac{\sin C_1}{\sin B_2} = \frac{\sin K_4}{\sin K_3} \text{ (similarly)},$$

$$= \frac{\sin K_5}{\sin K_6} = \frac{\sin K_8}{\sin K_7} \text{ (similarly)}.$$

Thus *EKG*, *FKH* are straight lines, and *AKC*, *BKD* are the bisectors of the angles they form. E. H. L.

1815. PLATO-MATION.

In analogue computers numerical magnitudes are represented by phsyical quantities : digital computers deal with numbers direct.—W. Sluckin, *Minds and Machines* (Penguin Books, 1954). [Per Mr. R. A. Fairthorne.]

1816. But even the learned Father Mersenne, whose books about music are among the most important documents of the 1630's, ends his discussion of the lute with :

A lute-player can do anything he pleases with his instrument. For instance he can demonstrate the geometrical and harmonic means, the squaring of the circle, the proportionate movements of the heavens and the celestial bodies or of the speeds of falling weights : all these and a thousand other things.—Thurston Dart, *The Interpretation of Music* (1954), p. 114. [Per Mr. R. B. Harvey.]

1817. *Mrs. Malaprop*. Observe me, Sir Anthony. I would by no means wish a daughter of mine to be a progeny of learning ; I don't think so much learning becomes a young woman ; for instance, I would never let her meddle with Greek, or Hebrew, or algebra, or simony, or fluxions, or paradoxes, or such inflammatory branches of learning—neither would it be necessary for her to handle any of your mathematical, astronomical, diabolical instruments.—But, Sir Anthony, I would send her, at nine years old, to a boarding school, in order to learn a little ingenuity and artifice. Then, sir, she should have a supercilious knowledge in accounts ;—and as she grew up, I would have her instructed in geometry, that she might know something of the contagious countries ;—but above all, Sir Anthony, she should be mistress of orthodoxy, that she might not misspell, and mispronounce words so shamefully as girls usually do ; and likewise that she might reprehend the true meaning of what she is saying. This, Sir Anthony, is what I would have a woman know ;—and I don't think there is a superstitious article in it.—R. B. Sheridan, *The Rivals*, Act I, scene 2. [Per Mr. J. C. W. De la Bere.]

SEVEN REGION MAPS ON A TORUS

By John Leech

It is well known that any map on a torus can be coloured with seven colours so that no two contiguous regions are of the same colour, and that fewer than seven colours will not suffice as there exist maps of seven regions of which each one abuts on to each other. In this paper all such maps of seven regions are analysed and a simple method of constructing them is given.

Since each region of the map abuts on to all six others, each region has at least six sides, and it follows at once from Euler's Theorem (which for the torus is $F - E + V = 0$) that each has exactly six sides, so that any two regions meet in a simple arc. The surface of a torus can be mapped onto part of a plane, for instance a rectangle, in such a manner that by bringing into coincidence its opposite sides the torus is obtained. If the plane is filled with a lattice of rectangles congruent to this rectangle, the regions of the map will map into a hexagonal network filling the plane. This network can be distorted into a regular hexagonal network; by this distortion the rectangles will in general not remain rectangular, but since they all undergo the same distortion, each will deform into a " curvilinear parallelogram ", i.e. a region bounded by a simple closed curve whose boundary may be divided into four parts, each being congruent and equivalent by translation to that opposite to it. The four points dividing the boundary are vertices of a (normal) parallelogram.

Let us now investigate how this regular network of hexagons can be coloured. Each region of one colour abuts on to six regions, one of each other colour, and these form a greater region shown in Figure 1. As each

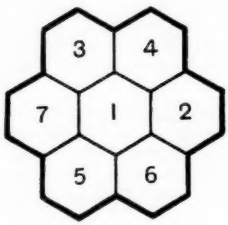


Fig. 1.

region of each other colour abuts on to just one region of the first colour, each hexagon is contained in just one of these greater regions, so that the plane is filled with these greater regions without overlapping. As further any translation of the network which transforms any curvilinear parallelogram into any other also transforms these greater regions into each other, they are similarly placed. These regions, however, can be fitted together in precisely two ways to fill the plane, namely as shown in Figure 2 and in an enantiomorphous arrangement which is otherwise equivalent. There are therefore two enantiomorphous colourings of the hexagonal network corresponding to colourings of the seven region maps on the torus.

In order to derive from this network the seven region maps on the torus, we note that the vertices of the curvilinear parallelogram correspond to the same point on the torus, so that they must therefore be equivalent under the translations of the network into itself, and that the curvilinear parallelogram has an area of seven hexagons. Thus if we set up a system of oblique Car-

tesian coordinates whose integer points are exactly all the points of the network corresponding to some point of it (which may be chosen arbitrarily, corresponding to an arbitrary point on the torus), we may choose as vertices of the parallelogram the origin, any integer point (p, q) whose coordinates are coprime, and any two consecutive integer points on one of the lines $qx - py = \pm 1$. Equivalent maps are obtained whichever of the four vertices of the parallelogram is chosen as origin, also if the parallelogram is rotated through $\frac{1}{3}\pi$ or $\frac{2}{3}\pi$, also whichever of the two vertices adjacent to the origin is taken as (p, q). In all, each map may thus be derived in 24 ways. There is a triply infinite set of topologically different seven region maps on the torus. It will be noted that not only does the choice of origin not matter in deriving the maps, but also the shape of the curvilinear sides of the parallelogram, since deforming these (without moving their ends) corresponds only to deforming the curves on the torus along which it was dissected.

The simplest choice for the vertices of the parallelogram is four corresponding points in the greater regions of Figure 2. Many of the seven region maps given in the literature* can be obtained by suitable choice of vertices and curvilinear sides for the region and suitable distortion of the parallelogram.

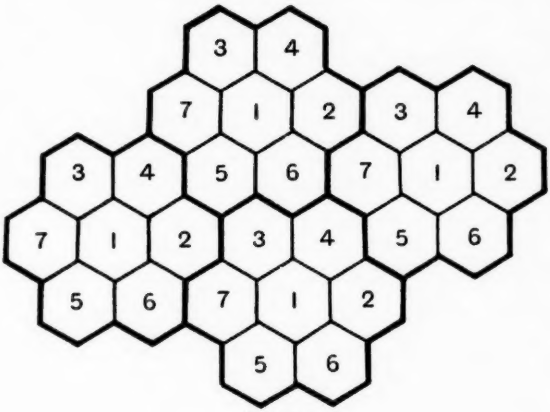


Fig. 2.

Seven Region Maps in which the Regions are Congruent

It is clear from the above construction that the seven regions on the torus are topologically equivalent, and it is of interest to investigate the circumstances in which they are geometrically congruent. When they are congruent, there is a transformation of the torus into itself which transforms

* e.g. D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie* (Berlin, 1932), p. 296, (English translation, *Geometry and the Imagination* (New York, 1952), p. 335); W. W. Rouse Ball, *Mathematical Recreations and Essays* (Revised Coxeter, London, 1939), p. 235; H. Martyn Cundy and A. P. Rollett, *Mathematical Models* (Oxford, 1952), p. 149; P. Ungar, "On Diagrams Representing Maps", *J. London Math. Soc.*, 28 (1953), p. 342, first diagram. His second diagram is equivalent to the first diagram in Figure 3 of this paper.

any one of the regions into any other. If the torus is the ordinary anchor ring formed by rotating a circle about a line in its plane not meeting it, the only such transformations are rotations of $2n\pi/7$ about the line for $n = 1, 2, 3, 4, 5, 6$. The seven regions being equivalent under such rotations, it follows that if the surface is mapped onto a rectangle whose sides correspond to a generating circle and a circle described by a point under the generating rotation, using the obvious conformal mapping which maps the two systems of circles into perpendicular systems of straight lines, the regions will be equivalent under translations in the direction of the latter sides and, having undergone the same distortion, will be congruent. To construct such maps by the present method, we have to construct suitable parallelograms in the network of hexagons. This may be done by choosing a point corresponding to the origin in the same relative position as it but to a hexagon of a different colour and taking as the point (p, q) the nearest integer point to the origin on the line joining the origin to this point just chosen. This side of the parallelogram will then contain points corresponding to the origin relatively to hexagons of each of the seven colours. Two maps constructed in this way are given in Figure 3. The networks are distorted only by parallel projection in this figure ; in the first, the diagram looks simpler if the zigzag line along the centre of the diagram is distorted into a straight line.

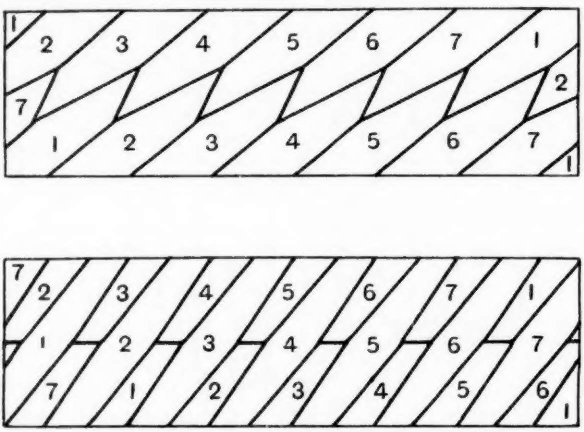


Fig. 3.

If instead of an anchor ring we take a rotating ring of tetrahedra,* we have various possibilities. The surface of such a ring can be unfolded into a parallelogram without distortion so there is no difficulty in making the regions of the map congruent as regards the surface. It is of greater significance to make the regions equivalent relatively to the tetrahedra ; in this case there must be some multiple of seven tetrahedra, and if the ring is cut along its edges to unfold into a parallelogram—this is the usual way of making these rings—the map on this parallelogram has to satisfy a similar congruence condition to that for the anchor ring. (As the ends of the ring may be joined so that the vertices at either end of the parallelogram coincide with the mid-point of the other end, the full period of the map may be twice the

* As suggested by J. M. Andreas (quoted by Coxeter in Rouse Ball, *loc. cit.*).

length of the parallelogram.) Two simple maps using some of the edges of the tetrahedra may be constructed as follows. Of the ring of seven tetrahedra, the unhinged edges form three closed polygonal lines on the surface which are geodesics. Two of these spiral once round the ring in one sense, the other spirals twice round the ring in the other sense like the edge of a Möbius ribbon. Choose on one of these lines fourteen equally spaced points and number them consecutively. Join each even-numbered point to that numbered five greater (modulo 14) by means of a geodesic which does not cross or coincide with the polygonal line, the seven joins being congruent. (There are two such joins in each case according to which way round the inside of the ring the geodesic goes relatively to the polygonal line. One is shorter and so more convenient.) To make a ring of seven tetrahedra fully rotatable, it is necessary for the unhinged edges to be rather more than half as long again as the hinged edges. The resulting models are particularly fascinating.

J.L.

1818. Your Correspondent, Mr. H. J. Finn, states perfectly correctly that the rate of cooling from boiling water is higher than that for cold.

If we assume that it takes say 30 minutes for a pound of boiling water to cool through the first 162 degrees down to 50 degrees and three hours for the remaining 18 degrees to freezing point, surely the boiling water starts with a thirty-minute—or 162 degrees handicap, for the cold water was at the 50 degrees temperature to begin with.

No. I would suggest that the reasons for this paradox are three-fold.

1. Physiochemical—in that cold water, especially in Coventry, contains a large amount of salts (which can be seen in quantity on the bottoms of kettles). Salt depresses the freezing point of water—that's why we put it on the roads. Boiling water loses much of its salts and hence freezes more nearly at 32 degrees Fahrenheit.

2. Physical—in that hot water especially if thrown on the ground or used to wash a flat surface, evaporates readily, leaving less water to freeze.

3. Psychological—in that we don't expect hot water to freeze quickly.

* * * * *

It takes some time for news to reach us here, but it has been brought to my notice that some doubt has been expressed in the columns of your eminent newspaper about my Law of Cooling.

When I first enunciated the law, the observations confirmed that the rate of cooling was proportional to the difference in the temperature between the cooling body and its surroundings. This was true for water as for all other liquids, and unless Coventry's water has the most peculiar physical properties, I would imagine that it too would obey the same law.

If my law, as stated in the differential form, is integrated, it will be seen, after a little elementary algebra, that the time taken for hot water to freeze completely differs from that for an equal volume of colder water, by an amount proportional to the difference of the logarithms of their original temperatures, provided, of course, that the surrounding temperature is below 0°C.

Recently, we had the pleasure of a visit of a person from " Another Place." It took him far longer to cool down to our ambient temperature than, I trust, it would your correspondent Mr. Finn.

Yours gravitationally,

ISAAC NEWTON.

Heaven.

Letters to *The Coventry Evening Telegraph*, February 12, 1954.

TRIANGLES WITH COMMON CIRCUMCENTRE AND ORTHOCENTRE

By D. G. Taylor.

1. The object of this paper may be best explained by reference to the figure. ABC is a triangle with circumcentre O and orthocentre P. What other triangles share these points with ABC?

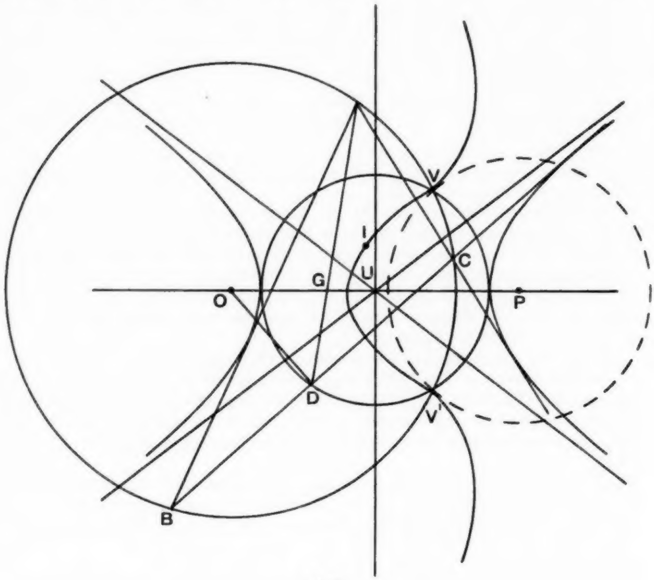


Fig.

There is clearly a doubly infinite family of such triangles. If we confine attention for the moment to those possessing the same circumradius R, we shall have a singly infinite family, which we may call *the family R*. These also possess in common, in addition to circumcentre and orthocentre,

(i) nine-points centre U (the middle point of OP) and nine-points circle, radius $\frac{1}{2}R$;

(ii) centroid G, where $(OGUP)$ is harmonic.

Putting $OP = h$, we know that

$$h^2 = R^2(1 - 8 \cos A \cos B \cos C), \qquad \text{.........................(1)}$$

and three cases arise.

I. $R < h$. All triangles are obtuse-angled; the circumcircle and nine-points circle cut in real points V, V'; the circle of centre P and radius $PV = 2R\sqrt{(-\cos A \cos B \cos C)}$ is real, and is polar for the family R; that is, it is such that each triangle of the family is self-conjugate with respect to it.*
This is the case illustrated in the figure.

* See, e.g., Lock and Child, *A new trigonometry for schools and colleges*, (Macmillan, 1911) p. 367.

II. $R > h$. All triangles are acute-angled; the points V, V' and the polar circle are imaginary.

III. $R = h$. The circumcircle and nine-points circle touch at P; all triangles are right-angled, with their right-angle vertices coincident at P; and the polar circle becomes a point circle at P.

Given h, R has no upper limit; but it has a lower limit, $\frac{1}{3}h$, approached as one angle of the triangle approaches the limit π.

2. It will be shown that:

(i) the sides of all triangles of the family R touch a conic of which O, P are the foci, and thus the triangles constitute a poristic family; also, as R varies while h remains constant, these conic envelopes (clearly) constitute a confocal system;

(ii) the incentre and excentres of all triangles of the family R lie on a Cartesian curve of which O is a singular focus and U an ordinary focus.

Further results will be obtained by reciprocation.

3. Since the nine-points circle bisects BC at D, and OD is at right angles to BC, it follows that BC is tangent to the conic of which O is a focus and the nine-points circle the auxiliary circle; and the other focus is clearly P. This argument applies to CA, AB, and so to the sides of all triangles of the family R. In case I, the conic is a hyperbola; in case II, an ellipse; in case III it degenerates into the point-pair O, P. Since O, P are fixed, the system of conics as R varies is confocal.

4. If we put $OI = r$, $UI = r'$, where I is the incentre of ABC, it is well known that

$$r^2 = 2Rr', \qquad\qquad (2)$$

and the same formula holds for the excentres I_1, I_2, I_3. This is the dipolar equation of a Cartesian curve of which O is a singular focus and U an ordinary focus. This Cartesian, then, is the locus of the incentres and excentres of all triangles of the family R. Also variation of R leads to a family of such Cartesians.

5. A few properties of the Cartesian (2) may be noted. With O as origin, and x-axis along OP, the equations of the circumcircle, nine-points circle and polar circle are

$$\left.\begin{aligned} S_O &\equiv x^2 + y^2 - R^2 = 0, \\ S_U &\equiv (x - \tfrac{1}{2}h)^2 + y^2 - \tfrac{1}{4}R^2 = 0, \\ S_P &\equiv (x - h)^2 + y^2 - \tfrac{1}{2}(h^2 - R^2) = 0, \end{aligned}\right\} \qquad (3)$$

where

$$2S_U - S_O = S_P.$$

The equation of the Cartesian takes the form

$$(x^2 + y^2)^2 = 4R^2\{(x - \tfrac{1}{2}h)^2 + y^2\}.$$

(i) It follows* that, when $R > h$, there are three ordinary foci on OP at distances from O

$$\tfrac{1}{2}h, \quad R\{R \pm \sqrt{(R^2 - h^2)}\}/h.$$

(ii) The curve crosses the x-axis where

$$x = R \pm \sqrt{\{R(R - h)\}} \quad \text{or} \quad -R \pm \sqrt{\{R(R + h)\}}.$$

When $R > h$, this yields four real points, separated by O, U, P, and indicating two ovals. When $R < h$, there is only one oval; and when $R = h$, the curve reduces to a limaçon with a node at P.

* Hilton, *Plane algebraic curves*, (Oxford, 1920), p. 319.

(iii) The form

$$(x^2+y^2-2R^2)^2 = -4R^2h\{x-(4R^2+h^2)/4h\}$$

of the equation puts in evidence the real double tangent.

(iv) Another form is

$$(x^2+y^2-R^2)^2 = 2R^2\{(x-h)^2+y^2-\tfrac{1}{2}(h^2-R^2)\},$$

or say,

$$t_1^2 = \sqrt{2}\,.\,Rt_2,$$

where t_1, t_2 are the lengths of the tangents from (x, y) to the circumcircle and polar circle respectively. This shows that the Cartesian passes through V, V' and touches the polar circle at these points.

In the figure the polar circle is indicated by a broken line, and part only of the Cartesian is drawn.

6. There is still another aspect of the Cartesian. Consider the tangents from A to the confocals of the family of which O, P are the foci. These tangents constitute an involution pencil, of which AII_1, I_2AI_3 are the double lines. There are double lines likewise at B and C; and I, I_1, I_2, I_3 are points at which these lines meet in threes. Hence the Cartesian is the locus of points of concurrence of double lines of the involution pencils determined at A, B, C by tangents to the conics of the confocal family, for all triangles ABC of the family R.

7. Let the configuration be reciprocated with respect to the polar circle S_P. Each triangle of the family R is transformed into itself. The circumcircle and hyperbola-envelope transform each into the other. The family of confocal conics changes into the family of coaxial circles of which P is a limiting point and the circumcircle S_O a member; so that the second limiting point is the inverse of P with respect to S_O.

The involution pencil of tangents from A to the confocal conics becomes the involution range cut on BC by the coaxial circles, and the double lines AII_1, I_2AI_3 become the double points of the range. This pair of points together with the corresponding pairs on CA, AB, constitute six points which lie, by threes, on four lines, the reciprocals of I, I_1, I_2, I_3. Hence the aggregate of such lines for all triangles of the family R envelopes a curve which is the reciprocal of the Cartesian (2) with respect to the polar circle.

D. G. T.

1819. How Much is Nothing?

Mr. J. G. Holt . . . said that the analyses of two medicines showed one to consist of 99·96 p.c. sugar and the other 99·94 p.c. sugar. There was nothing else.—*Daily Telegraph*, October 21, 1954. [Per Mr. Harry E. Airey.]

1820. The shopkeeper was selling an electric computing machine, but he reckoned the bill on a bead abacus.—James Cameron (*News-Chronicle* correspondent on a visit to Peking). (*News-Chronicle*, Nov. 15, 1954). [Per Mr. E. H. Lockwood.]

1821. The Human Robot.

At a recent meeting of air scientists and pilots, the scientists made it clear that they sould like to replace the pilot in the aircraft with instruments and servo-mechanisms. Scott Crossfield, a famous test pilot, rejoined by asking "Where can you find another non-linear servo-mechanism weighing under eleven stone and having great adaptability than can be produced as cheaply by unskilled labour?"—*New York Times*, quoted in the *Reader's Digest*, September 1954. [Per Mr. B. M. Brown.]

SOME CONSIDERATIONS OF GRAVITY

By R. C. Lyness.

The centre of mass, G, of a system of particles (masses m_r at P_r) can be defined by the relation, O being an arbitrary point,

$$(\Sigma m_r)\,\mathbf{OG} = \Sigma(m_r\,.\,\mathbf{OP}_r). \quad \text{.............................(i)}$$

This is equivalent to finding the point Q_1 which divides P_1P_2 in the ratio $m_2 : m_1$; then the point Q_2 which divides Q_1P_3 in the ratio $m_3 : (m_1 + m_2)$; then the point Q_3 which divides Q_2P_4 in the ratio $m_4 : (m_1 + m_2 + m_3)$, and so on until finally G is reached. For simple coordinate geometry shows that, whatever order we take the masses in, this process produces a cartesian coordinate $\bar{x}$ for $G(=Q_{n-1})$ given by

$$(\Sigma m_r)\bar{x} = \Sigma(m_r x_r), \quad \text{.................................(ii)}$$

and this is the projection of (i) on the (arbitrary) x-axis.

For most purposes we can regard a body as being made up of particles on which the earth exerts parallel forces proportional to their masses. These parallel forces (called the weights of the particles) are equivalent to a resultant force which, by the rule of combination of parallel forces, acts through the centre of mass of the particles. When we are thinking of the resultant of the weights of the particles it is usual to call the centre of mass the "centre of gravity" of the body.

Now what happens when we replace the assumption that the weights of the particles form a system of *parallel* forces by the assumption that the lines of action of the forces act through the centre of the earth?

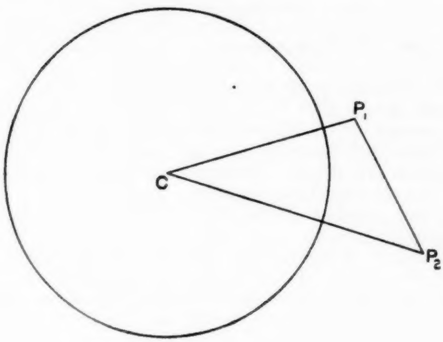


Fig. 1

Let us simplify the position by considering only two particles and first taking their weights to be proportional to their masses. We must then find the resultant of two forces of magnitudes km_r acting along P_rC, $(r = 1,\ 2)$. Writing $P_1C = r_1$ and $P_2C = r_2$ these forces can be represented by the vectors

$$(km_1/r_1)\,.\,\mathbf{P_1C} \quad \text{and} \quad (km_2/r_2)\,.\,\mathbf{P_2C},$$

and are thus equivalent to

$$k(m_1/r_1 + m_2/r_2)\,.\,\mathbf{QC},$$

where Q divides P_1P_2 in the ratio $(m_2/r_2) : (m_1/r_1)$. The centre of mass of m_1 and m_2 divides P_1P_2 in the ratio $m_2 : m_1$, and hence the resultant of the weights of the two particles does not pass through their centre of mass unless they are equidistant from the earth's centre.

In fact a more realistic assumption is that the force exerted by the earth on a particle is proportional not only to its mass but also to the inverse square of its distance from the centre. On this assumption we have to find the resultant of

$$(\gamma M m_1/r_1^3) \,.\, \mathbf{P_1C} \quad \text{and} \quad (\gamma M m_2/r_2^3) \,.\, \mathbf{P_2C}.$$

This passes through the point which divides P_1P_2 in the ratio $(m_2/r_2^3) : (m_1/r_1^3)$; again, not through the centre of mass.

It is thus evident that a body cannot generally be said to have a centre of gravity. Under the two assumptions considered, when we turn the body round, the line of action of the earth's resultant attraction does not pass through any fixed point of the body. If, however, the gravitational force is directly proportional to the distance from the centre of attraction the same argument as given above shows that the resultant attraction *does* pass through the centre of mass, which can then be truly called the centre of gravity of the body.

If we were enclosed in a hollow sphere and were uncertain whether we were at the bottom of the sea or above the earth's surface, the mere rotating of a uniform rod about a horizontal axis through its middle point would decide the issue. For beneath the earth's surface the earth's attractive force varies directly as the distance from the centre and so the resultant of the weights of the particles will pass through the middle point of the rod. Above the earth's surface where the inverse square law holds, the resultant weight does not pass through the middle point except when the rod is horizontal and so it will have a moment about the axis. The angular velocity of the rod when it is set spinning will remain constant in the first case but will vary in the second. The detecting of the presence or absence of this variation would dispel our initial uncertainty.

We have made the assumption that the earth attracts as if it were a particle of the same mass situated at its centre. What theoretical justification is there for this assumption? Let us assume first that the earth is a sphere of uniform density and that every particle attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. We will find the resultant of the forces which all the particles of the sphere exert on a particle of unit mass outside the sphere.

Consider a particle P of the sphere, mass δm, distant r from the particle of unit mass at O. The force it exerts on O is $(\gamma \,.\, \delta m/r^3)\mathbf{OP}$, where γ is the constant of gravitation. Summing for all those particles of the sphere which are distant r from O and thus lie on a spherical cap, we have a resultant force of $(\gamma m/r^3)\mathbf{OG}$, where G is the mass centre of the spherical cap and m its mass. By considering the circumscribing cylinder of Archimedes, the area of the cap is $2\pi r(r-x)$ and $OG = \frac{1}{2}(r+x)$, x being the distance of O from the point of intersection of OC with the plane through the circular edge of the cap.

Now consider particles whose distances from O are $r + \theta \,.\, \delta r$, $0 \leqslant \theta \leqslant 1$. The ratio of their total mass to $2\pi\rho r(r-x) \,.\, \delta r$, where ρ is the density, tends to 1 as δr tends to 0. Their total attraction is directed along OC and the ratio of its magnitude to

$$\frac{2\pi\gamma\rho r(r-x)\ \delta r}{r^3} \,.\, \tfrac{1}{2}(r+x)$$

tends to 1 as δr tends to 0. Summing, the whole sphere exerts an attraction of

$$\pi\rho\gamma\int_{c-a}^{c+a}\left(1-\frac{x^2}{r^2}\right)dr\,.$$

Now $2cx = c^2 + r^2 - a^2$, so this is an easy integral to evaluate, particularly if we simplify by putting $c+a=p$, $c-a=q$ and $2cx = pq + r^2$. Its value is $4\pi\gamma\rho a^3/3c^2$, which is $\gamma M/c^2$, where M is the mass of the sphere. This proves the result.

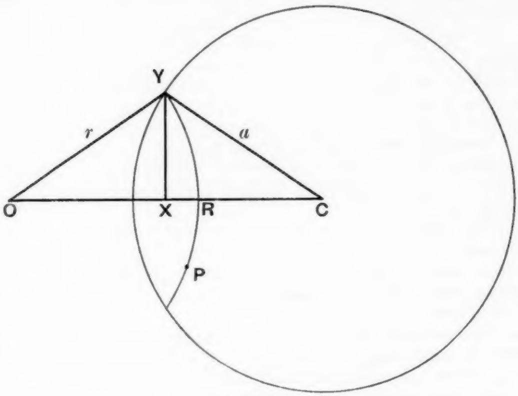


FIG. 2.

$OX = x$, $OC = c$, $OY = OR = OP = r$, $CY = a$.

If O is inside the sphere, that is if $c < a$, the total attraction of the particles within a sphere centre O and radius $a - c$ is zero and the limits of integration are from $a - c$ to $a + c$. The total attraction is $4\pi\gamma\rho c/3 = \gamma Mc/a^3$, and is thus directly proportional to the distance of the unit mass from the centre.

We can now remove the restriction that the sphere should be of uniform density, for the attraction of a spherical shell of uniform density, of internal radius a and external radius b, on unit mass outside the shell is

$$4\pi\gamma\rho b^3/3c^2 - 4\pi\gamma\rho a^3/3c^2 = \gamma m/c^2,$$

where m is the mass of the shell. It follows by addition that the attraction on external unit mass of a sphere whose density varies but is the same at all points equidistant from the centre is still $\gamma M/c^2$.

The attraction of a spherical shell on unit mass *inside* the shell is

$$4\pi\gamma\rho c/3 - 4\pi\gamma\rho c/3 = 0.$$

It follows that the total attraction on unit mass distant c from the centre, where $c < a$, is $\gamma M'/c^2$, where $M' = 4\pi\rho c^3/3$. This is equal to $\gamma Mc/a^3$ and so the restriction to uniform density can be replaced by density which is the same at all points equidistant from the centre, whether the attracted mass is outside or inside the sphere. The assumption of variable density is much more reasonable physically than that of uniform density throughout the sphere.

It is natural to consider whether the attraction of some other shaped body under the inverse square law is the same as if its mass were concentrated at the centre. If we consider the ellipsoid obtained by rotating the ellipse

$$(x-c)^2/a^2 + y^2/b^2 = 1$$

about the x-axis we get as before a total attraction of

$$\pi\rho\gamma\int_{c-a}^{c+a}\left(1-\frac{x^2}{r^2}\right)dr$$

but now
$$\frac{(x-c)^2}{a^2}+\frac{r^2-x^2}{b^2}=1$$

and upon substitution the indefinite integral contains a non-algebraic term $k\theta$, where $r^2(a^2-b^2)=b^2(b^2+c^2-a^2)\sinh^2\theta$. This is enough to show that the formula for the magnitude of the resultant force, $\gamma M/c^2$, which holds for a sphere, does not hold for an ellipsoid. The line of action of the resultant passes through the centre, by symmetry, for the unit mass lies on the axis of revolution. I do not see how to prove it but I conjecture that as the unit mass moves on a circle centre C and radius c in a plane through the axis the line of action envelopes a curve and does not always pass through the centre of the ellipsoid.

Another question which naturally arises is whether a law of attraction other than the inverse square would still make a sphere attract as if its mass were concentrated at its centre.

From first principles a law directly proportional to distance gives a resultant attraction through the mass-centre of the body whatever its shape or position relative to the unit mass attracted. For any particle P_r of the body attracts with a force $km_r\,.\,\mathbf{P}_r\mathbf{O}$ and the sum of these forces is $k(\Sigma m_r)\mathbf{CO}$, where C is the mass-centre of m_r at P_r.

That a law of attraction $kr+\gamma/r^2$ will make the total attraction of a sphere the same as if its mass were concentrated at its centre follows by considering the two terms of the law separately and adding the two resultant forces. That this is the *only* law which gives the result has been proved recently (see *American Mathematical Monthly*, October 1951, Vol. 58, No. 8, p. 571). The proof involves mathematics normally outside the scope of the Sixth Form, but a sixth-former may easily verify that other laws of attraction do not give the same simple result as does the inverse square law. For instance, the total attraction of a sphere under an inverse cube law is, when $c>a$,

$$\pi\rho\gamma\int_{c-a}^{c+a}\left(\frac{r^2-x^2}{r^3}\right)dr$$

which is equal to

$$\frac{3\gamma M}{16a^3c^2}\left\{-4ac+2(c^2+a^2)\log\frac{c+a}{c-a}\right\}.$$

When a/c is small, this is approximately

$$\frac{\gamma M}{c^3}\left(1+\frac{2a^2}{5c^2}\right).$$

R. C. L.

1822. Query.—Will you please confirm to some acquaintances of mine that by multiplying ANY number or numbers by nought, then the answer is nought—i.e. 17,000 × 0 = 0?

Answer.—Sufferin' cats! Did you ever hear such nonsense! Listen, woman. Suppose you take out of your purse six pennies, see. Shove 'em on the table and then multiply 'em by nothing. Do you expect to find the six pennies wafted into thin air, leaving nothing? Use your noddle. If you multiply a number by nought, there's no change in the number.—*Daily Mirror*, February 24, 1953. [Per Mr. C. Birtwistle.]

31-POINT GEOMETRY

By W. L. Edge.

The following paragraphs have been assembled in consequence of my reading Dr. Cundy's note on 25-point geometry.* Towards the end of it, apparently mindful of the adjunction of a " line at infinity " to the Euclidean plane, he adjoins a line to the 25-point plane and so obtains a geometry of 31 points. Here I reverse this procedure : I start with the 31-point geometry and thereafter assign to one of its 31 lines the rôle of the " line at infinity ". This seems more in the spirit of Cayley's dictum at the end of his Sixth Memoir on Quantics, that " descriptive geometry is *all* geometry " and metrical geometry only a part thereof.

Finite geometries arise whenever coordinates are confined to a finite field. Any such field consists of p^n " marks ", p being a prime number, and the resulting plane geometry consists of $p^{2n}+p^n+1$ points. If $p^n=5$ we obtain the 31-point geometry and in it, as we shall see, harmonic properties are prominent.

1. The integers, positive, negative and zero, may be distributed among five residue classes

$$0,\ 1,\ -1,\ 2,\ -2$$

to modulus 5. Each integer belongs to one and only one of the classes, which are added and multiplied commutatively according to the modulus 5 so that, for example

$$1+2=-2, \qquad (-2)(-2)=-1.$$

Moreover each non-zero class has a unique inverse :

$$1\,.\,1=-1\,.\,-1=2\,.\,-2=-2\,.\,2=1.$$

The classes indeed form a *field* F, and we call them the *marks* of F. In F 1 and -1 each have two square roots, those of 1 being 1 and -1 and those of -1 being 2 and -2. But 2 and -2 have no square roots in F.

If x_1 and x_2 are two marks we can, save when $x_1=x_2=0$, regard them as homogeneous coordinates of a point of a line L. The point is unaltered if both x_1 and x_2 are multiplied by the same non-zero mark, so that L consists of $24\div 4=6$ points. Whenever $x_2\neq 0$ we label the point by the mark $x=x_1/x_2$; when $x_2=0$ we use the sixth mark ∞.

There are projectivities

$$axx'+bx+cx'+d=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

among the points of L, such a projectivity being uniquely determined when three corresponding pairs of points x, x' are assigned. The number of projectivities is therefore ${}^6P_3=120$; they form a group, triply transitive on the points of L and subjecting them to 120 permutations. It is not, in general, possible to fit a projectivity on to four pairs, but we do have, for any four distinct points A, B, C, D,

$$ABCD \barwedge DCBA \barwedge CDAB \barwedge BADC. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

It is of course understood that the coefficients a, b, c, d are all marks of F.

2. Any four points of L have cross ratios, and cross ratio is invariant under projective transformation. But since every cross ratio of four among

$$0,\ 1,\ -1,\ 2,\ -2,\ \infty$$

is itself among these six marks the cross ratios of four distinct points on L can only be -1, 2, -2 ; the other marks 0, 1, ∞ occur as cross ratios only when two of the four points coincide.

* *Math. Gazette* 36 (1952) 158–166.

The transposition of any pair of the four points cannot, without the simultaneous transposition of the complementary pair, be brought about by a projectivity; it permutes the three cross ratios and, being of period 2, must transpose two of them and leave the third unchanged. The transposition of the complementary pair of points imposes the same permutation of the cross ratios. There are three ways of separating the four points into complementary pairs and each of the three cross ratios is unchanged by the transposition of either pair for one of the three separations; the separation for which -1 is the invariant cross ratio is *harmonic separation*, and *every set of four distinct points on L admits a harmonic separation.* This is surely the most significant feature of the geometry. We say, as usual, that either pair is harmonically conjugate to the other, and the condition for A, B and C, D to be harmonically conjugate is

$$(x_A + x_B)(x_C + x_D) = 2(x_A x_B + x_C x_D). \qquad (2.1)$$

We may also say that A and B are harmonic inverses of one another in C and D, or that A and B are a pair of the involution whose foci are C and D. If harmonically conjugate pairs on L are given by quadratics

$$a_1x^2 + 2h_1x + b_1 = 0, \qquad a_2x^2 + 2h_2x + b_2 = 0$$

then

$$a_1b_2 + a_2b_1 = 2h_1h_2, \qquad (2.2)$$

it being understood that ∞ is a root of a " quadratic " wherein x^2 has coefficient zero. Any quadratic whose roots are both marks of F has its coefficients in F too although, as $x^2 = 2$ exemplifies, the converse is not true. We cannot, for instance, as we could in the field of complex numbers, assert that two pairs determine a unique pair harmonic to them both: an involution need not have real foci.

The projectivity 1.1 is an involution when $b = c$, and the foci of

$$axx' + b(x + x') + d = 0$$

are the roots of

$$ax^2 + 2bx + d = 0 \qquad (f)$$

These belong with a, b, d to F whenever $b^2 - ad$ is a square in F; but should $b^2 - ad$ be either of the non-squares ± 2 the foci are " conjugate imaginaries ". In this event neither a nor d can be zero, for this would involve the contradiction that the square of b was a non-square; we may therefore suppose, without affecting the roots of (f), that $a = 1$ and

$$d = b^2 \pm 2.$$

The mark b can be any one of the five; once b has been chosen there are two values for d, and so there are ten involutions whose foci are not " real ". On the other hand any two points of L determine an involution of which they are the foci, so that there are fifteen involutions whose foci are " real ". These are the harmonic inversions, each point of L having, by 2.1, a unique harmonic conjugate in regard to any two given points.

3. Take any pair A, B of points on L; the remaining four points are thereby separated into pairs C, D and E, F each of which is harmonic to A, B. Not only so: C, D and E, F are harmonic to each other. For the harmonic conjugate of C in regard to E and F is distinct from C, E, F and cannot be A or B each of which has the other for its harmonic conjugate; it must therefore be D. Each of the fifteen pairs of points induces such a separation of the six into three mutually harmonic pairs so that there are $15 \div 3 = 5$ such *sextuples*, the term sextuple implying the pairing and harmonic separations. A sextuple is indeed what Sylvester called a syntheme, but only five of the fifteen synthemes are sextuples. Since every pair determines one and only one sextuple

no two sextuples have a pair in common ; the five sextuples together constitute what Sylvester called a synthematic total. It is one, T call it, of the six such totals that arise from the six points. Any projectivity on L leaves invariant cross ratio and the harmonic relation, and therefore T ; and the fact that no projectivity can change T into any other of the six totals accords with the fact that only one sixth of the 6! permutations of the points of L can be achieved by projectivities. We now display the sextuples of T, with the quadratics whose roots are the mutually harmonic pairs ; any two quadratics in the same row satisfy 2.2. Once a sextuple is given the others are obtainable from it by projectivities ; indeed by relations $x' = x + m$ where m belongs to F. Thus the remaining sextuples of T follow at once from the one in the top row.

$\infty,\ 0$	$1,\ -1$	$2,\ -2$	x	x^2-1	x^2+1
$\infty,\ 1$	$2,\ 0$	$-2,\ -1$	$x-1$	x^2-2x	x^2-2x+2
$\infty,\ 2$	$-2,\ 1$	$-1,\ 0$	$x-2$	x^2+x-2	x^2+x
$\infty,\ -2$	$-1,\ 2$	$0,\ 1$	$x+2$	x^2-x-2	x^2-x
$\infty,\ -1$	$0,\ -2$	$1,\ 2$	$x+1$	x^2+2x	x^2+2x+2

4. If A, B ; C, D ; E, F is a sextuple then

$$ABED \barwedge ABFC \barwedge BACF, \qquad \text{(4.1)}.$$

the first projectivity being harmonic inversion in A and B and the second the simultaneous transposition of complementary pairs in accordance with 1.2 This shows that AB, CE, DF are in involution, and we prove in this manner that the three pairs of any syntheme extraneous to T are in involution. These are the ten involutions with " conjugate imaginary " foci.

These ten involutions may also be obtained as follows. Take any three distinct points X, Y, Z on L and let

X' be the harmonic inverse of X in Y and Z,
Y' be the harmonic inverse of Y in Z and X,
Z' be the harmonic inverse of Z in X and Y.

All the points of L are hereby accounted for. Now the sextuple determined by the pair YZ must be

$$Y, Z \qquad X, X' \qquad Y', Z'$$

so that X is the harmonic inverse of X' in Y' and Z'.
Likewise Y is the harmonic inverse of Y' in Z' and X'
and Z is the harmonic inverse of Z' in X' and Y'.

The complementary triads XYZ and $X'Y'Z'$ are similarly related, and every pair of complementary triads bear this relation to each other. Moreover, just as in 4.1,

$$XX'Y'Z \barwedge XX'Z'Y \barwedge X'XYZ'$$

so that XX', YY', ZZ' are in involution. The foci of this involution cannot be among the points of L since all six are accounted for by the three pairs, and the number of such involutions is ten, the number of pairs of complementary triads. In this notation the synthematic total of sextuples is

$$\begin{array}{ccc} YZ & XX' & Y'Z' \\ ZX & YY' & Z'X' \\ XY & ZZ' & X'Y' \\ XY' & YZ' & ZX' \\ XZ' & YX' & ZY' \end{array}$$

and the ten synthemes extraneous to it are each composed of three pairs in involution.

This geometry, wherein a line consists of six points that can be separated in five ways into mutually harmonic pairs while each of the other ten synthemes yields pairs in involution, is very briefly, though quite explicitly, alluded to by Fano in his paper " Sui postulati fondamentali della geometria proiettiva " in volume 30 of *Giornale di Matematiche* (1892); see therein page 123.

5. Geometry can be based on F for a space of any number of dimensions. In a plane the number of points is the number, 5^3-1, of triplets (x_1, x_2, x_3) of marks not all of which are zero, divided by the number, $5-1$, of non-zero marks. This yields a 31-point geometry; there are 31 points lying 6 on each of 31 lines, each point lying on 6 lines. Similarly we can set up a three dimensional geometry of 156 points, and so on. In all these spaces every set of four distinct collinear points admits a harmonic separation. These geometries are all self-dual. In the plane for example a line answers to a triplet (u_1, u_2, u_3) of marks which are not all zero, and line and point are incident whenever $u_1x_1+u_2x_2+u_3x_3=0$.

6. We now consider the geometry in such a plane Π, and first obtain the numbers of polygons. In this context the use of such a word as triangle, — —, hexagon implies automatically that no three vertices are collinear.

There are 25 points not on the join of two given points and 16 not on any side of a given triangle; hence the number of quadrangles is

$$31 \cdot 30 \cdot 25 \cdot 16 \div 4! = 15500.$$

We can commence the construction of a pentagon by taking one of the quadrangles, whose four vertices and three diagonal points are then not eligible for the remaining vertex of the pentagon. Neither are any of the other three points on any of the six sides of the quadrangle (each side contains two vertices and one diagonal point). The number of points which are eligible is therefore

$$31-4-3-6 \cdot 3=6$$

and so the number of pentagons is $15500 \times 6 \div 5 = 18600$.

If a pentagon is to be amplified, by choosing a further vertex, to a hexagon we are debarred from choosing any of its five vertices, fifteen diagonal points or any of the other points one on each of the ten sides (each side containing two vertices and three diagonal points). The number of eligible vertices is therefore

$$31-5-15-10=1$$

so that the number of hexagons is $18600 \times 1 \div 6 = 3100$.

7. Let α, β be any two vertices of a hexagon $h \equiv \alpha\beta\gamma\delta\epsilon\zeta$. There are four other points on $\alpha\beta$ and the six sides of the quadrangle $\gamma\delta\epsilon\zeta$ must each pass through one of them. Hence $\alpha\beta$ contains two diagonal points of $\gamma\delta\epsilon\zeta$, say Q, the intersection of $\delta\epsilon$ and $\gamma\zeta$, and R, the intersection of $\gamma\epsilon$ and $\delta\zeta$. The intersections Y, Z of QR with $\epsilon\zeta$ and $\gamma\delta$ are harmonic to Q, R and α, β complete the sextuple. Thus on each *side* s of h, and constituting a sextuple thereon, are two *vertices* V, two *diagonal points* D and two *Brianchon points* B, these last being points of concurrence of three s that join the six V in pairs. Since there are fifteen s, two through each D and three through each B, h has in all fifteen D and ten B which, together with its six V, account for all the points of Π.

Of the six lines through any vertex V_0 five are sides s; they account for all ten B, two of which are on each s, and all the other V, but only for ten of the fifteen D. The remaining five D therefore lie on the one remaining line through V_0. Such a line, consisting of one V and five D, we call a *tangent* t;

all these names are of course related to h and a point or line is differently named in relation to different hexagons. There are six t, one associated with each V, forming a hexagram H. Any two of them meet in a D and the fifteen D are the intersections of the pairs of t just as the fifteen s are the joins of the pairs of V.

We have now seen that when two V α, β have been chosen two of the three separations of the other four into complementary pairs yield pairs of s whose intersection is a B on $\alpha\beta$. The third pairing however does not: $\alpha\beta$, $\gamma\delta$, $\epsilon\zeta$ are not concurrent but form a triangle XYZ each of whose vertices is a D. Each side of XYZ determines the other two, just as $\alpha\beta$ above determined $\gamma\delta$ and $\epsilon\zeta$, and each s determines one and only one such triangle. Hence there are five of these triangles: they answer to the five synthemes of V in one synthematic total. The vertices of the triangles account for all the D, their sides for all the s.

8. Take any D, for example the intersection X of $\gamma\delta$, $\epsilon\zeta$. Through it pass the s XY, XZ each containing two V, two B and one D other than X. There are also two t, $X\alpha$ and $X\beta$, each containing four D other than X and one V. These two pairs of lines are harmonic. The third pair XQ, XR of the sextuple has to account for the remaining four D and six B. Neither line contains any V and we call this third type of line a Pascal line p; each p contains three D (hence its name) and three B, and since two p pass through each D there are ten p altogether. The three B, one on each of $\alpha\beta$, $\gamma\zeta$, $\delta\epsilon$ apart from their point of concurrence Q, are collinear for they are, respectively, R, the harmonic conjugate of Q in regard to γ and ζ, and the harmonic conjugate of Q in regard to δ and ϵ; the two latter points lie, by the harmonic property of $\gamma\delta\epsilon\zeta$, on RX, which is a p. The ten p are thus associated one with each B.

Since there are ten p with three of the ten B on each, each B lies on three p. Indeed the p could have been thus obtained. For through Q there pass three s which together account for six V, six D and three B other than Q. None of the three remaining lines through Q can contain a V and they have together to account for six B (other than Q) and nine D.

If we delete the three p through Q and the one, q call it for the moment, that is associated with Q there remain six p passing two through each of the three B on q. Thus the ten p and ten B form a Desargues figure.

9. Since each of the 3100 hexagons distributes the 31 lines of π as six t, ten p and fifteen s it follows that any given line L of Π is a tangent of 600, a Pascal line of 1000 and a side of 1500 hexagons.

10. We may define a conic Σ in Π as the set of intersections of corresponding rays of two related pencils, with the proviso that the join $\alpha\beta$ of their vertices is not self-corresponding. The pencils can be related in ${}^6P_3 = 120$ ways, the relation being determined when to three fixed lines through α correspond three distinct lines through β. But in ${}^5P_2 = 20$ of these relations $\alpha\beta$ and $\beta\alpha$ correspond, so that there are only 100 conics through α and β. Likewise there are ${}^5P_2 - {}^4P_1 = 16$ conics circumscribing a triangle and ${}^4P_1 - 1 = 3$ circumscribing a quadrangle. There is a unique conic circumscribing a pentagon. Indeed a conic consists, as a locus, simply of the V of a hexagon h; the t are *tangents* of Σ and the s are *chords* of Σ. For of the six lines through α the one which, in the pencil with vertex α, corresponds to $\beta\alpha$ does not meet Σ elsewhere while each of the other five meets Σ once elsewhere, namely at its intersection with the corresponding line through β. The fifteen D, each the intersection of two t, are *external* to Σ and are the *poles* of the fifteen s; the D and s are the vertices and sides of five *self-polar triangles*. None of the ten B can lie on any t; they are *internal* to Σ. A second kind of self-polar triangle, of which there are fifteen, has one vertex external and the other two both internal to Σ.

11. There are on Σ involutions and harmonic pairs, sections of involutions

and harmonic pairs of lines of any pencil whose vertex is on Σ. Thus any four points α, β, γ, δ of Σ admit one separation, say $\alpha\beta$, $\gamma\delta$ into harmonic pairs. Each of $\alpha\beta$, $\gamma\delta$ passes through the pole of the other since

$$\alpha(\alpha\beta\gamma\delta)\barwedge\beta(\alpha\beta\gamma\delta)\barwedge\beta(\beta\alpha\gamma\delta),$$

a projectivity between the pencils with vertices α, β in which $\alpha\beta$ is self-corresponding, so that γ, δ are collinear with the pole of $\alpha\beta$. The chords $\alpha\beta$, $\gamma\delta$ are *conjugate*, and $\epsilon\zeta$ is conjugate to both of them ; the three chords form a self-polar triangle whose vertices are all D. We see again how the five triangles of this kind answer to the five synthemes of V in one synthematic total.

The involution whose foci are α and β has γ, δ and ϵ, ζ as two pairs ; it is cut on Σ by the lines through the pole X of $\alpha\beta$, although the two p through X do not meet Σ. There are also involutions centred at the points B. For example, in the figure already used for h,

$$\alpha(\alpha\beta\gamma\delta)\barwedge XZ\gamma\delta\barwedge XY\epsilon\zeta\barwedge\alpha(\alpha\beta\epsilon\zeta),$$

the projection from XZ on to XY being from R, so that, on Σ,

$$\alpha\beta\gamma\delta\barwedge\beta\alpha\zeta\epsilon$$

and the concurrent chords $\alpha\beta$, $\gamma\zeta$, $\delta\epsilon$ do give the pairs of an involution. There are ten involutions of this kind on Σ, one centred at each B.

12. Pascal's Theorem has to hold for Σ since it is a consequence of the projective generation. But while six points of general position on a conic give rise, in projective geometry over the field of complex numbers, to sixty Pascal lines, here only the ten p are eligible. It was Veronese who discovered that the general figure of sixty Pascal lines is composed of six Desargues figures; here a single Desargues figure serves six times over. That each p arises for six different orderings of the V is clear. For let the V be separated into two complementary triads in any one of the ten possible ways, and use the notation of § 4 ; the six orderings

$$\begin{array}{lll} XX'YY'ZZ' & XY'YZ'ZX' & XZ'YX'ZY' \\ XYZX'Y'Z' & YZXY'Z'X' & ZXYZ'X'Y' \end{array}$$

are found all to yield the same p, opposite sides of any one of these hexagons always intersecting on this p.

13. It scarcely needs saying that the statements dual to all those in §§ 6–12 also hold, the geometry in Π being self-dual. Σ may be regarded not only as the assemblage of its six points but also as the assemblage of its six tangents. It can be generated by a projective relation between any two of its tangents in which their intersection does not correspond to itself.

14. Choose now any one, L say, of the 31 lines in Π as absolute line. Any two lines which meet on L we call parallel, and the remaining 30 lines are distributed in six parallel sets of five. We have a geometry, when L is omitted, of 25 points and 30 lines, six lines passing through each point and five points being on each line, and the completely symmetrical duality so evident in Π is now destroyed. The mid-point of $\alpha\beta$ is the harmonic conjugate, in regard to α and β, of the intersection of $\alpha\beta$ with L ; given any three collinear points (none of them on L) some one of them bisects the join of the other two.

Those 600 conics of which L is a tangent are parabolas ; they fall into six families of 100, all of any one family having parallel axes.

Those 1500 conics of which L is a chord are hyperbolas; they fall into fifteen families of 100, all of any one family having parallel asymptotes. The four points on any hyperbola are vertices of a parallelogram whose diagonals are diameters, meeting at the centre of the curve. Of the 100 members of any family four have their centre at any one of the 25 points not on L; this is merely equivalent to saying that, given any two points α, β on L and a point C not on L there are four projectivities between the pencils whose vertices are α, β in which $\alpha\beta$, αC correspond respectively to βC, $\beta\alpha$.

Those 1000 conics of which L is a Pascal line are ellipses; they may be separated into ten families of 100 by using the ten involutions on L (cf. § 4) with conjugate imaginary foci. The B which is the pole of L is the centre of the ellipse, whose six points lie two on each of three diameters.

15. Let us now take any one of these ten involutions and decree that its foci, I and J, be the absolute points. Their coordinates are not marks of F so that they do not belong to Π any more than Poncelet's points belong to the real Euclidean plane; but they may serve as absolute points just as Poncelet's do. Let the pairing of points on L in the involution, be

$$X, X'; \quad Y, Y'; \quad Z, Z';$$

each pair is harmonic to I and J. And now there are six directions distributed as three perpendicular pairs as Dr Cundy so perspicuously describes. The hyperbolas of three of the fifteen families are now rectangular and the ellipses of one of the ten families are now circles.

16. We add an instance or two of the use of coordinates. Recall that, as laid down in § 1,

(i) All coordinates x, y, z are marks of F and any marks serve so long as all three coordinates are not zero simultaneously,

(ii) (x, y, z) and (mx, my, mz) are the same point for any non-zero mark m.

If a conic circumscribes the triangle of reference its equation is

$$fyz + gzx + hxy = 0$$

where f, g, h all belong to F and, if the conic is presumed non-degenerate, $fgh \neq 0$. We may therefore take $f = 1$ without affecting the conic; there remain four choices for each of g and h so that, as the projective generation made clear in § 10, there are sixteen conics circumscribing a triangle.

The six points on $yz + zx + xy = 0$ are

1, 0, 0	0, 1, 0	0, 0, 1
2, 1, 1	1, 2, 1	1, 1, 2.

When arranged in this manner in three pairs their joins are concurrent at (1, 1, 1), which is therefore a Brianchon point. The fifteen D are the intersections of pairs of the six tangents

$$\begin{array}{lll} y + z = 0 & z + x = 0 & x + y = 0 \\ y + z = x & z + x = y & x + y = z. \end{array}$$

Three of these D lie on $x + y + z = 0$, which is a Pascal line.

A similar discussion shows that there are sixteen non-degenerate conics for which a given triangle is self-polar.

The six points on $x^2 + y^2 + z^2 = 0$ are those whose coordinates are permutations of (0, 1, 2). The fifteen D, arranged as vertices of five self-polar triangles, are

1, 0, 0	2, 1, 1	1, 2, 2	2, − 1, 1	2, 1, − 1
0, 1, 0	1, 2, 1	− 1, 2, 1	2, 1, 2	1, 2, − 1
0, 0, 1	1, 1, 2	− 1, 1, 2	1, − 1, 2	2, 2, 1

and the ten B are

$$\begin{array}{ccccc} 0,1,\ \ 1 & 1,0,1 & 1,\ \ 1,0 & -1,1,\ \ 1 & 1,-1,1 \\ 0,1,-1 & -1,0.1 & 1,-1,0 & 1,1,-1 & 1,\ \ 1,1. \end{array}$$

$x+y+z=0$ is a Pascal line for this conic too, containing

$$\begin{array}{rccc} \text{three } B: & 0,1,-1 & -1,0,1 & 1,-1,0 \\ \text{and three } D: & 1,2,\ \ 2 & 2,1,2 & 2,\ \ 2,1 \end{array}$$

but for the former conic $yz+zx+xy=0$ the rolès of these six points are reversed, the three former then being D and the three latter B. But if we choose $x+y+z=0$ as absolute line both conics are ellipses with centre $(1, 1, 1)$.

17. We are able to retain symmetry in the three coordinates by selecting I and J to satisfy

$$x^2+y^2+z^2=x+y+z=0,$$

whereupon the above two ellipses become concentric circles. Indeed all the conics

$$x^2+y^2+z^2=k(x+y+z)^2$$

with k a mark of F are circles with centre $(1, 1, 1)$ except when $k=2$; the discriminant then vanishes and

$$x^2+y^2+z^2-yz-zx-xy=0$$

is a point-circle. This equation may, among other forms, be written

$$(x+2y+2z)^2+2(y-z)^2=0,$$

and since 2 is not a square both linear forms appearing here must be zero simultaneously for the equation to hold.

More generally: the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

cannot be a circle unless

$$a+2f=b+2g=c+2h,$$

when its equation is

$$a(x^2-yz)+b(y^2-zx)+c(z^2-xy)+d(yz+zx+xy)=0$$

and represents a circle provided that its discriminant is not zero. The coordinates of the centre are

$$2a+b+c+2d, \qquad a+2b+c+2d, \qquad a+b+2c+2d.$$

Thus the circles with centre (A, B, C) have equations

$$(2A+B+C)(x^2-yz)+(A+2B+C)(y^2-zx)+(A+B+2C)(z^2-xy)$$
$$+d(2x^2+2y^2+2z^2-yz-zx-xy)=0.$$

The discriminant of this quadratic form is

$$\Delta \equiv (A+B+C)\{(A+B+C)d+A^2+B^2+C^2-BC-CA-AB\}.$$

It is implied that $A+B+C$ is not zero, and the four genuine circles occur for those four marks d for which $\Delta \neq 0$.

Notes and References.

1. Sylvester introduces synthemes of six objects in his paper "Elementary Researches in the Analysis of Combinatorial Aggregation"; this is on p. 91 of Vol. I of his *Mathematical Papers* and he gives an etymology for his neologism in a footnote on this page. On p. 92 he displays a synthematic total: i.e. a set of five synthemes of which the pairs, three in each syntheme, together

exhaust all ${}^6C_2 = 15$ pairs. Given any one syntheme two totals can be built to include it, wherefore the number of distinct totals is $15 \times 2 \div 5 = 6$.

2. When we adjoin to F either root of any one of those ten quadratics (f) whose discriminant is a non-square we thereby generate a larger field Φ of twenty-five marks. If the roots of $x^2 = 2$ are $\pm j$ each of the ten quadratics has a pair of roots $A \pm Bj$ where A is a mark and B a non-zero mark of F. These twenty roots, with the five marks of F, constitute Φ. Just as there are primitive marks, namely ± 2, in F of which all non-zero marks are powers, so in Φ; the first twenty-four powers of, for example, $2 - j$ yield all the non-zero marks of Φ. The one-dimensional geometry based on Φ consists of the six points of L and the pairs of foci of the ten involutions, 26 points in all.

3. Veronese's long account of the Hexagrammum Mysticum of Pascal is in vol. I of the third series of Memoirs of the *Atti della Reale Accademia dei Lincei* (1877); pp. 649–702. The separation into six Desargues figures is on p. 661. It was on reading the manuscript of this memoir before its publication that Cremona realised how to obtain the whole Pascal figure by projection from a node of a cubic surface. Concerning these matters see H. W. Richmond: *Mathematische Annalen* 53 (1899), 161–176.

4. Cayley's Sixth Memoir on Quantics was published in vol. 149 of the *Philosophical Transactions* in 1859 and occupies pp. 561–592 of vol. II of his *Collected Mathematical Papers*. It does not presume a knowledge of its five predecessors and should be read by every mathematician. It is a pleasure, as well as an education, to read it and Forsyth says, in his obituary notice of Cayley, that it could not be presented in more attractive form.

The choice as absolute of a pair of points was made in § 15 in order to derive the geometry described by Dr. Cundy. But it is clear from the Memoir that we might equally well have chosen as absolute any of the 3100 conics and so obtained a geometry that is "non-euclidean" but in which the duality between points and lines is still symmetrical.

5. Given a quadrangle in Π there is a unique projectivity transforming its vertices into those of the same or any other quadrangle taken in any prescribed order. The group G of projectivities in Π is therefore of order 372000, 4! times the number of quadrangles. G is transitive on the lines and on the conics of Π.

Any non-singular three-rowed matrix M imposes, whenever its nine elements all belong to F, one of these projectivities, but the same projectivity is imposed by all four matrices $\pm M$, $\pm 2M$. The number of non-singular matrices is therefore 1488000, four times the number of projectivities; these matrices form the general linear homogeneous group (on three variables, over F) whose order is given in the classical treatises, for instance on p. 97 of C. Jordan's *Traité des substitutions* (Paris 1870) and on p. 77 of L.E. Dickson's *Linear Groups* (Leipzig 1901). From these sources we derive the order 1488000 in the form $(5^3-1)(5^3-5)(5^3-5^2)$. Since the multiplication of M by a mark m of F causes the determinant $|M|$ to be multiplied by m^3 we can set up a (1, 1) correspondence between matrices and the projectivities which they impose by stipulating that $|M|$ is always 1, and we thus obtain G as the special linear homogeneous group of order $(5^3-1)(5^3-5)5^2$.

Any conic Σ is invariant for $372000 \div 3100 = 120$ projectivities of G. Since, by choice of the triangle of reference, Σ has the equation $x^2+y^2+z^2=0$, the 120 unimodular matrices constitute the orthogonal group R (in three variables and over F). This is of the same order as the group, encountered in § 1, of projectivities on L; indeed the two groups are not merely of the same order but are isomorphic, both of them being symmetric groups of degree 5. The projectivities on L permute five of six synthematic totals while R permutes those five self-polar triangles of Σ whose vertices are all D.

W. L. E.

I

SOME SUMMATION FORMULAE FOR BINOMIAL COEFFICIENTS

By T. Narayana Moorty.

Dr. S. Vajda commences Note 2159 (*Math. Gazette*, Vol. 34, p. 211) with the following :

" In his review of H. S. M. Coxeter's *Regular Polytopes* (*Math. Gazette* Vol. 33, p. 49) Mr. H. Martyn Cundy quotes the formula

$$\sum_{s=0} \left(-\tfrac{1}{4}\right)^s \binom{n-s}{s} = (n+1)/2^n. \qquad \text{.........................(1)}$$

and supplies the proof by considering the coefficients of x^n in the expansion of

$$\frac{1}{1-x}\left[1+\frac{\frac{1}{4}x^2}{1-x}\right]^{-1} = (1-\tfrac{1}{2}x)^{-2}$$

Ingenious as this certainly is, it suffers in the present writer's mind from the disadvantage that it makes the result appear a curiosity rather than a particular case of a theorem of wide application. I venture therefore to give the following derivation of this and other results."

With my rather inadequate resources, I make bold to show below that the method of comparing coefficients is powerful in addition to being ingenious. To this end, I give below a proof by this good old method of Dr. Vajda's generalisation of (1) :

$$\sum_{s=0} t^s \binom{n-s}{s} = \frac{1}{\alpha-\beta}\left[\alpha^{n+1}-\beta^{n+1}\right], \qquad \text{.........................(2)}$$

where $\alpha = \frac{1}{2}\left[1+(1+4t)^{1/2}\right]$, and $\beta = \frac{1}{2}\left[1-(1+4t)^{1/2}\right]$ ....(3)

Incidentally, this method is used to generalise (2) further and to obtain a number of new and interesting results.

2. Consider the identity

$$\frac{1}{1-x}\left[1-\frac{tx^2}{1-x}\right]^{-1} = (1-x-tx^2)^{-1} \qquad \text{.........................(4)}$$

The right side can be taken as $[(1-\alpha x)(1-\beta x)]^{-1}$ if $t \neq -\frac{1}{4}$, where α, β are given by (3). This leads us to the partial fractions

$$\frac{1}{\alpha-\beta}\left[\frac{\alpha}{1-\alpha x}-\frac{\beta}{1-\beta x}\right]$$

for the right side.

Comparing coefficients of x^n in the expansions on either side of (4) we have Dr. Vajda's generalisation (2).

3. (2) can be further generalised by considering the identity

$$\frac{1}{1-x}\left[1-\frac{tx^r}{1-x}\right]^{-1} = (1-x-tx^r)^{-1} \qquad \text{.........................(5)}$$

Putting $f(x) = 1-x-tx^r$,

we have $f'(x) = -(1+rtx^{r-1})$,

and $f''(x) = -r(r-1)tx^{r-2}$,

Obviously $f'(x)$ and $f''(x)$ cannot have a common root, so that $f(x)$ can have, if at all, a root repeated only twice. Also the repeated root should satisfy the equation

$$rf(x) - xf'(x) = 0$$

which gives $$x = r/(r-1).$$
This gives $$t = -(r-1)^{r-1}/r^r.$$
as the condition that $f(x)=0$ may have a root repeated twice. In all other cases $f(x)=0$ has distinct roots.

When the roots, say $\alpha_1, \alpha_2 \dots \alpha_r$ are all distinct,

$$\frac{1}{f(x)} = \frac{-1}{t(x-\alpha_1)\dots(x-\alpha_r)} = \sum_r \frac{1}{f'(\alpha_r)} \cdot \frac{1}{x-\alpha_r}.$$

Hence comparison of coefficients of x^n on either side of (5) gives

$$\sum_{s=0} \binom{n-(r-1)s}{s} t^s = -\sum_r \frac{1}{f'(\alpha_r)} \cdot \alpha_r^{-(n+1)} \quad \text{for } t \neq -(r-1)^{r-1}/r^r \qquad \text{.......................(6)}$$

which is a generalisation of (2). The case of repeated root's can be similarly dealt with.

4. It is interesting to note that the result (6) is obtained by either of the following methods also :

(*a*) Start from the identity

$$\frac{1}{1-tx^r}\left[1 - \frac{x}{1-tx^r}\right]^{-1} = (1-x-tx^r)^{-1} \qquad \text{.......................(7)}$$

and proceed as in para 3.

(*b*) Expand $[1-(x+tx^r)]^{-1}$

as $$\sum_p x^p(1+tx^{r-1})^p = \sum_{p,s} x^p \binom{p}{s} t^s x^{(r-1)s},$$

and equate the coefficient of x^n so obtained with that obtained by the method of partial fractions.

Method (*b*) implies that (1) can be established by equating coefficients of x^n in

$$[1-(x-\tfrac{1}{4}x^2)]^{-1} = (1-\tfrac{1}{2}x)^{-2}.$$

5. Another set of new and interesting results can be obtained from the identity :

$$\frac{1}{1-x}\left[1 - \frac{tx+x^r}{1-x}\right]^{-1} = [1-(t+1)x-x^r]^{-1}. \qquad \text{..................(8)}$$

Proceeding as in para 3, we can show that the equation

$$\phi(x) = 1-(t+1)x-x^r = 0$$

has one root repeated twice and the other $r-2$ roots distinct when

$$(t+1)^r = -r^r/(r-1)^{r-1}$$

and has r roots distinct for all other values of t.

In either case the right side of (8) can be put into partial fractions.

When the roots $\alpha_1, \alpha_2, \dots \alpha_r$ are all distinct,

$$\frac{1}{\phi(x)} = \frac{-1}{(x-\alpha_1)\dots(x-\alpha_r)} = \sum_r \frac{1}{\phi'(\alpha_r)} \cdot \frac{1}{x-\alpha_r}.$$

The coefficient of x^n on R. S. is thus

$$-\sum_r \frac{1}{\phi'(\alpha_r)} \cdot \alpha_r^{-(n+1)}.$$

The L.S.

$$= \mathop{\Sigma}_{s}(tx + x^r)^s(1-x)^{-(s+1)}$$

$$= \mathop{\Sigma}_{s,p,q} \binom{s}{p} t^{s-p} \,.\, x^{rp+s-p} \,.\, \binom{s+q}{s} x^q.$$

The coefficient of x^n in this is obtained by summing over all s, p, q satisfying $s+q+(r-1)p=n$, and so

$$= \mathop{\Sigma}_{s,p} \binom{s}{p} \binom{n-(r-1)p}{s} t^{s-p}.$$

Thus, we have,

$$\mathop{\Sigma}_{s,p} \binom{p}{s} \binom{n-(r-1)p}{s} t^{s-p} = -\mathop{\Sigma}_{r} \frac{1}{\phi'(\alpha_r)} \,.\, \alpha_r^{-(n+1)} \quad \text{..............(9)}$$

for $\qquad (t+1)^r \neq -r^r/(r-1)^{r-1}$

The partial fractions and the value in the exceptional case can be obtained by a slightly different method.

6. The R.S. of (2) can be taken as

$$\frac{1}{2^{n+1}}[(1+\gamma)^{n+1}-(1-\gamma)^{n+1}]/\gamma \quad \text{where} \quad \gamma=(1+4t)^{1/2},$$

$$= \frac{1}{2^n} \mathop{\Sigma}_{p} \binom{n+1}{2p+1} \gamma^{2p} = \frac{1}{2^n} \mathop{\Sigma}_{s,p} \binom{n+1}{2p+1} \binom{p}{s} 4^s t^s$$

Equating coefficients of t^s in this and the L.S. of (2), we have

$$\mathop{\Sigma}_{p} \binom{n+1}{2p+1} \binom{p}{s} = 2^{n-2s} \binom{n-s}{s} \,. \quad \text{..........................(10)}$$

Substituting this in (1) we get

$$\mathop{\Sigma}_{s,p} (-1)^s \binom{n+1}{2p+1} \binom{p}{s} = n+1. \quad \text{..........................(11)}$$

7. Starting from

$$\frac{1}{1+x+x^2}\left[1-\frac{t}{1+x+x^2}\right]^{-1} = [(1-t)+x+x^2]^{-1} \quad \text{..............(12)}$$

it can be shown that

$$\mathop{\Sigma}_{p,s} (-1)^s \binom{p+1}{n-3s} \binom{p+s}{s} t^p = (n+1) \,.\, 2^{n+2}, \quad \text{if } t=\tfrac{3}{4}\,; \quad \text{......(13a)}$$

$$= \frac{(-1)^n}{1-t} \left(\frac{a^{n+1}-b^{n+1}}{a-b} \right), \text{ if } t \neq \tfrac{3}{4}, \quad \text{(13b)}$$

where $a, b = \{-1 \pm \sqrt{(4t-3)}\}/2(1-t)$.

8. From

$$\frac{1}{1+x+x^2}\left\{1-\frac{tx}{1+x+x^2}\right\}^{-1} = \{1+(1-t)x+x^2\}^{-1} \quad \text{..............(14)}$$

we get

$$\mathop{\Sigma}_{p,s} (-1)^{n+s+p} \binom{p+1}{n-3s-p} \binom{p+s}{s} t^p = n+1, \quad \text{if } t=3, \quad \text{................(15a)}$$

$$= (-1)^n(n+1), \quad \text{if } t=-1, \quad \text{...(15b)}$$

$$= \frac{c^{n+1}-d^{n+1}}{c-d} \quad \text{otherwise} \quad \text{.......(15c)}$$

where $c, d = \tfrac{1}{2}\{t-1 \pm \sqrt{(t^2-2t-3)}\}$.

Expanding R.S. of (14) as $[1+\{(1-t)x+x^2\}]^{-1}$ and equating coefficients of x^n, we have

$$\sum_{s,p}(-1)^{n+s+p}\binom{p+1}{n-3s-p}\binom{p+s}{s}t^p=\sum_{s}(-1)^{n-s}\binom{n-s}{s}(1-t)^{n-2s} \quad \ldots (16)$$

Equating coefficients of t^p on either side of (16)

$$\sum_{s}(-1)^{s+p}\binom{n-s}{s}\binom{n-2s}{p}=\sum_{s}(-1)^{s+p}\binom{p+1}{n-3s-p}\binom{p+s}{s} \quad \ldots\ldots\ldots\ldots (17)$$

Putting $t=1$ in (15) we find that $c, d=\pm i$. And so,

$$\sum_{s,p}(-1)^{s+p}\binom{p+1}{n-3s-p}\binom{p+s}{s}=\begin{cases}(-1)^{n/2}, & \text{if } n \text{ even}\\ 0, & \text{if } n \text{ odd}\end{cases} \quad \ldots\ldots\ldots\ldots (18)$$

9. Similarly from

$$\frac{1}{1+x+x^2}\left[1-\frac{tx^2}{1+x+x^2}\right]^{-1}=[1+x+(1-t)x^2]^{-1} \quad \ldots\ldots\ldots\ldots (19)$$

we get

$$\sum_{s,p}(-1)^s\binom{p+1}{n-3s-2p}\binom{p+s}{s}t^p=\begin{cases}(n+1)/2^n, & \text{if } t=\frac{3}{4}\\ (-1)^n\left[\dfrac{e^{n+1}-f^{n+1}}{e-f}\right] & \text{otherwise}\end{cases} \quad \ldots (20)$$

where $e, f=\frac{1}{2}[-1\pm(4t-3)^{\frac{1}{2}}]$.

Adopting the procedure of para 6 to the R.S. of (20) we have,

$$2^{n-2p}\sum_{s}(-1)^s\binom{p+1}{n-3s-2p}\binom{p+s}{s}=\sum_{r}\binom{n+1}{2r+1}\binom{r}{p}(-3)^{r-p}. \quad \ldots (21)$$

10. Observing that a, b of para 7 and e, f of para 9 are connected by the formulae $(1-t)(a,b)=(e,f)$ we have from (13) and (21)

$$(1-t)^{n+1}\sum_{s,p}(-1)^s\binom{p+1}{n-3s}\binom{p+s}{s}t^p$$
$$=\sum_{s,p}(-1)^s\binom{p+1}{n-3s-2p}\binom{p+s}{s}t^p. \quad \ldots\ldots\ldots\ldots (22)$$

11. I take this opportunity to give a new proof for a well-known result. Consider the geometric progression

$$1+(1+x)+(1+x)^2+\ldots(1+x)^{(n-1)}=\frac{(1+x)^n-1}{x}.$$

Equating coefficients of x^s, we have the known formula,

$$\sum_{p=0}^{n-1}\binom{p}{s}=\binom{n}{s+1}. \quad \ldots\ldots\ldots\ldots (23)$$

T. N. M.

1823. A queer little thing is the decibel,
Arithmetically quite unassecibel;
Multiply it by three
And what do we see?
An increase that's infinitecibel.

—Thomas Dalby, in *The Observer*, July 11, 1954. [Per Mrs. W. A. Fish.]

THE OSCILLATION OF A HEAVY SPRING

F. M. Arscott,

Introduction.

The longitudinal oscillation of "light" springs carrying loads, in various combinations, has a well-established place in all mechanics textbooks, but none of the standard works appear to deal thoroughly with the oscillation of a spring whose mass is not negligible. The purpose of this note is to investigate the vertical longitudinal oscillation of a heavy spring, first by itself and secondly when carrying a load. It will be seen that in each case the motion consists not of a single S.H.M. (as for the light spring), but an infinite number of S.H.M.s ; when the spring oscillates by itself the frequency of these S.H.M.s are multiples of a fundamental frequency, but when the spring carries a load the frequencies are not so simply related.

(The closely related problem of the oscillation of a bar is treated by operational methods in Carslaw and Jaeger's *Operational Methods in Applied Mathematics*, 2nd edition, pp. 138–41. This note, however, uses direct methods only and also includes approximations for the important case when the ratio of the mass of the spring to the mass of the load is small.)

We make the assumptions that (i) the spring when unstretched and horizontal is uniform—i.e. the mass per unit length is the same throughout, and (ii) each part of the spring obeys Hooke's Law with the same modulus of elasticity. We denote the natural length of the spring by a, the modulus of elasticity by λ and the mass per unit length, when horizontal and unstretched, by μ.

The equilibrium position.

Since the spring is heavy, when hung vertically from one end it will stretch under its own weight, and we must first find the form of the spring in equilibrium. It is convenient to take as "reference position", the spring as it would be if gravity did not act—i.e. vertical, but unstretched and uniform. Let this be the position OA, O being fixed and A the free end, and consider an element of the spring PQ, where $OP=x$, $PQ=\Delta x$. When gravity acts, let the whole spring take up the position OA' and P and Q move to positions P', Q', where $PP'=y$. $QQ'=y+\Delta y$. Let the tensions at P', Q' be T, $T+\Delta T$ respectively. Then, since the mass of the element $P'Q'$ is $\mu\Delta x$,

$$\Delta T+\mu g\Delta x=0.$$

Also, since the extension of the element is Δy,

$$T+\Delta T=\lambda\frac{\Delta y}{\Delta x}.$$

Proceeding to the limit as $\Delta x\to 0$, $$\frac{dT}{dx}=-\mu g, \qquad (1)$$

$$T=\lambda\frac{dy}{dx}. \qquad (2)$$

Hence y satisfies the equation $$\lambda\frac{d^2y}{dx^2}=-\mu g \qquad (3)$$

with the boundary conditions

$$y=0 \text{ when } x=0 \text{ and } \frac{dy}{dx}=0 \text{ when } x=a \qquad (4a)$$

(since clearly the tension vanishes at the lower end).

Solving:
$$y=\frac{\mu g}{2\lambda}(2ax-x^2). \qquad \text{(5a)}$$

This expression gives the displacement of a point on the spring, which in the "reference position" is a distance x from the fixed end, when gravity acts.

If, instead of the lower end being free, it carries a concentrated load M, we still have equations (1), (2), and (3), but instead of (4a) we have (by considering the equilibrium of the load M) the condition

$$\frac{dy}{dx}=\frac{Mg}{\lambda} \quad \text{when} \quad x=a. \qquad \text{(4b)}$$

This gives the solution

$$y=\frac{g}{2\lambda}((2M+2\mu a)x-\mu x^2). \qquad \text{(5b)}$$

The equation of motion.

Let the spring, loaded or unloaded, be disturbed so that it oscillates longitudinally, and at time t let the displacement of the point P of the spring from the equilibrium position P' be z (so that the displacement from the reference position is $y+z$). z is therefore a function of x and t. The motion is easily found to be governed by what is in effect the one-dimensional equation of wave motion

$$\frac{\partial^2 z}{\partial x^2}=\frac{\mu}{\lambda}\frac{\partial^2 z}{\partial t^2},$$

or, writing $c=\sqrt{(\lambda/\mu)}$ for brevity,

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}. \qquad \text{(6)}$$

The tension of the spring is given by

$$T=\lambda\frac{dy}{dx}+\lambda\frac{\partial z}{\partial x}, \qquad \text{(7)}$$

which can, of course, be simplified by substituting the appropriate expression for y from (5a) or (5b).

(For the method of obtaining equation (6) see, for example, Ramsey's *Hydrodynamics*, p. 322–4.)

The boundary and initial conditions which have to be satisfied in addition to (6) differ in the unloaded and loaded cases, and we consider these separately.

Solution for the unloaded spring.

We have first to obtain initial conditions, which will depend on the manner in which the spring is disturbed. The simplest manner, and that which we shall consider, is for the lower end of the spring to be pulled downwards a distance b from the equilibrium position and then released from rest. By considering the equilibrium of an element of the spring before it is released, with similar working to that which led to (5a), we find that

$$z=bx/a \text{ when } t=0, \text{ for all values of } x\leqslant a. \qquad \text{(8)}$$

Also, since the spring is released from rest,

$$\frac{\partial z}{\partial t}=0 \text{ when } t=0, \text{ for all values of } x\leqslant a. \qquad \text{(9)}$$

There are also the boundary conditions:

$$z=0 \text{ when } x=0 \text{ for all values of } t \geqslant 0, \quad \ldots\ldots\ldots\ldots(10)$$

and, since the tension vanishes at the lower end:

$$\frac{\partial z}{\partial x}=0 \text{ when } x=a \text{ for all values of } t \geqslant 0. \quad \ldots\ldots\ldots\ldots(11a)$$

Now (6) is known to be satisfied by any functions of the forms

$$\begin{matrix}\sin\\ \cos\end{matrix}(rx)\begin{matrix}\sin\\ \cos\end{matrix}(crt),$$

but to satisfy conditions (9) and (10) we must use only solutions of the type $\sin rx \cos crt$. Further, applying (11a) gives $\cos ra=0$, and hence

$$ra=(n+\tfrac{1}{2})\pi, \text{ or } r=(n+\tfrac{1}{2})\pi/a,$$

n being any integer.

We therefore assume a formal solution

$$z=\sum_{n=0}^{\infty} C_n \sin(2n+1)\pi x/2a \cos(2n+1)\pi ct/2a.$$

(Taking n through the range of negative integers introduces no essentially different term.)

To determine the constants C_n we apply the remaining condition (8) and find that

$$bx/a=\sum_{n=0}^{\infty} C_n \sin(2n+1)\pi x/2a.$$

Multiplying by $\sin(2n+1)\pi x/2a$, integrating from $-a$ to $+a$, and using the well-known results

$$\int_{-a}^{a} \sin(2n+1)\pi x/2a \sin(2n'+1)\pi x/2a\, dx=0, \quad (n \neq n'),$$

$$\int_{-a}^{a} \sin^2(2n+1)\pi x/2a\, dx=a,$$

$$\int_{-a}^{a} x\sin(2n+1)\pi x/2a\, dx=\frac{2(-1)^n a^2}{(n+\frac{1}{2})^2\pi^2},$$

we obtain

$$C_n=\frac{8b}{\pi^2}\frac{(-1)^n}{(2n+1)^2}.$$

The solution of the problem for the unloaded spring is therefore that the displacement from the equilibrium position at time t, after being stretched a distance b and released from rest, is:

$$z=\frac{8b}{\pi^2}\left[\sin\pi x/2a\cos\pi ct/2a-\frac{1}{9}\sin 3\pi x/2a\cos 3\pi ct/2a \right.$$
$$\left. +\frac{1}{25}\sin 5\pi x/2a\cos 5\pi ct/2a-\ldots\ldots\right]$$

where $c=\surd(\lambda/\mu)$.

The oscillation of the spring therefore consists of a fundamental oscillation of period $4a/c=4a\surd(\mu/\lambda)$ whose amplitude increases sinusoidally with x, (in the case of a light spring the amplitude increases proportionally to x) together with the third, fifth, ... , etc., harmonics of this oscillation.

It is noteworthy that the amplitudes of these harmonics not only fall off in the ratio $1/3^2$, $1/5^2$, ... , but that there are points of the spring which, relatively, are "nodes" for these particular harmonics—e.g. the points for which $\sin 5\pi x/a = 0$, i.e. the points for which $x = 2a/5$, $x = 4a/5$, do not move at all in the fifth harmonic.

In practice, μ/λ is generally so small that even the fundamental period is very small, and it would seem difficult to devise an experiment to verify these results.

Solution for the loaded spring.

If the spring carries a load M, the equation of motion is still (6); the conditions (8), (9) and (10) still apply, but instead of (11a) we have, from the motion of the load,

$$M\frac{\partial^2 z}{\partial t^2} = Mg - T \quad \text{when } x = a \text{ for all } t \geqslant 0.$$

Using (7) and recalling that y in this case is given by (5b) this becomes

$$M\frac{\partial^2 z}{\partial t^2} = -\lambda\frac{\partial z}{\partial x} \quad \text{when } x = a \text{ for all } t \geqslant 0. \quad \text{.................(11b)}$$

The solution $\sin rx \cos crt$ will satisfy this condition if

$$-Mc^2r^2 \sin ra = -\lambda r \cos ra,$$

i.e. $r \tan ra = \mu/M$ or $ra \tan ra = \mu a/M$.

But $\mu a =$ the total weight of the spring, and if we denote this by m, and m/M by k, then

$$ra \tan ra = k. \quad \text{....................................(12)}$$

The frequencies of the oscillations are determined by the roots of this equation, which is a transcendental one; we can get a good idea of the nature of these roots by drawing the graphs of $\tan ra$ and k/ra. There is an infinity of roots, but the positive and negative ones are numerically equal; one root lies in each of the ranges $s\pi < ra < (s+1)\pi$ (s being an integer) and this root becomes very nearly equal to $s\pi$ for large values of s. If we denote the positive roots of this equation by r_0, r_1, r_2, etc., then the solution of the problem of the spring is of the form

$$z = \sum_{n=0}^{\infty} C_n \sin r_n x \cos r_n ct. \quad \text{...........................(13)}$$

To evaluate the C_n we apply the condition (8) which gives:

$$bx/a = \sum_{n=0}^{\infty} C_n \sin r_n x. \quad \text{..............................(14)}$$

The series on the right is not a Fourier series; the constants could, however, be determined if we could find a set of functions $f_n(x)$ such that the integral of $f_n(x) \sin r_{n'} x$ vanished (over a certain range) when $n \neq n'$, while the integral of $f_n(x) \sin r_n x$ over the same range did not (i.e. a set of functions which are orthogonal to the functions $\sin r_n x$). There do not appear to be any simple functions forming such a set, but if we differentiate (14) with respect to x (assuming this to be legitimate) we obtain

$$b/a = \sum_{n=0}^{\infty} C_n r_n \cos r_n x. \quad (15)$$

Now it may easily be verified that if r_n and $r_{n'}$ are any two different positive

roots of the equation (12), then by straightforward integration and making use of (12) itself,

$$\int_0^a \cos r_n x \cos r_{n'} x \, dx = 0,$$

while
$$\int_0^a \cos^2 r_n x \, dx = a/2\left[1 + \frac{k}{(r_n a)^2 + k^2}\right]$$

and
$$\int_0^a \cos r_n x \, dx = \frac{\sin r_n a}{r_n}.$$

Multiplying both sides of (15) by $\cos r_n x$ and integrating from 0 to a, we obtain :

$$C_n = \frac{2b \sin r_n a}{(r_n a)^2\left[1 + \frac{k}{(r_n a)^2 + k^2}\right]}. \quad \text{.............................(16)}$$

The solution of the problem for the loaded spring is therefore given by (13), the values of the constants C_n being given (in terms of the r_n) by (16).

Approximate values for small k.

The case of most practical interest is that in which the mass of the spring is small compared with the load—i.e. $m/M = k$ is small. We can then obtain expressions for the r_n and hence the C_n in terms of k as follows :

Set $ra = \theta$, then equation (12) becomes

$$\theta \tan \theta = k, \quad \text{......................................(17)}$$

and let the positive roots of this be θ_0, θ_1, θ_2, etc., ($\theta_n = r_n a$).

Since θ_0 will be small if k is small, $\tan \theta_0 \simeq \theta_0$ and hence $\theta_0 \simeq \sqrt{k}$. If we now set $\theta_0 = \sqrt{k}(1 + \alpha k + \beta k^2)$ in (17), ignoring powers of k above the second we find $\alpha = -1/6$, $\beta = 11/360$, so that $\theta_0 = \sqrt{k}(1 - k/6 + 11k^2/360)$. Similarly $\theta_n \simeq n\pi$ ($n \geqslant 1$) and the corresponding closer approximation is

$$\theta_n = n\pi + k/n\pi - k^2/n^3\pi^3.$$

Substituting these values in (16) and continuing the same degree of approximation,

$$C_0 = \frac{b}{\sqrt{k}}(1 + k/3 - k^2/24),$$

$$C_n = 2b(-1)^n (k/n^3\pi^3 + 2k^2/n^5\pi^5). \quad \text{..........................(18)}$$

The periods of the various S.H.M.s are obtained from the r_n, where $r_n = \theta_n/a$.

The motion is dominated by the oscillation in the longest period, which has periodic time $T_0 = 2\pi/r_0 c$, which on substituting for r_0 and c becomes

$$T_0 = 2\pi\sqrt{(Ma/\lambda)}\,(1 + k/6 - k^2/360),$$

which equals
$$2\pi\sqrt{\{(Ma/\lambda)\,(1 + k/3 + k^2/45)\}}.$$

This may be written in the form

$$T_0 = 2\pi\sqrt{\{(a/\lambda)\,(M + m/3 + m^2/45M)\}} \,;$$

and since the period of a light spring carrying a load M' is $2\pi\sqrt{(M'a/\lambda)}$ we see the effect of the mass of the spring on the main oscillation is approximately the same as adding a mass $m/3 + m^2/45M$ to the concentrated load M ; this agrees with the first approximation which may be derived from physical considerations " add one-third of the mass of the spring to the mass carried ".

The amplitude of the main oscillation, being given by $C_0 \sin r_0 x$, varies sinusoidally with x, the amplitude of the oscillation of the load itself (which we denote by A_0) being given by $A_0 = C_0 \sin r_0 a = C_0 \sin \theta_0$. For k small this is given approximately by $A_0 = b(1 - 7k^2/180)$.

Similarly the period of the oscillation of frequency $r_n c$ is found to be, for small k,

$$T_n = (2/n)\sqrt{(ma/\lambda)}(1 - k/n\pi + 2k^2/n^4\pi^4) \quad (n \geqslant 1),$$

the corresponding amplitude A_n for the oscillation of the load being

$$2bk^2/n^4\pi^4.$$

These show that for small k, the diminution in amplitude of the main oscillation due to the mass of the spring is very small, while the amplitudes of the new oscillations which it introduces are extremely small. For example, if $k = 0{\cdot}1$, then $A_0 = 0{\cdot}9996b$, while $A_n = 0{\cdot}0002b/n^4$.

Note.

The expression for C_n given by (18) shows that, for small values of k at least, $C_n = 0(n^{-3})$; consequently the series on the right-hand side of (14) may be shown to be uniformly convergent for $x \leqslant a$, and the term-by-term differentiation by which we obtained (15) is legitimate.

F. M. A.

Makerere College,
The University College of East Africa,
Kampala, Uganda.

1824. Lessons on Maths.

(with apologies to Henry Reed and his " Naming of parts ")

To-day, we have square roots and surds ; yesterday
We had tangents and chords ; and to-morrow morning we
Shall have angles, sine, cos and tan ; but to-day
We have square roots and surds. Flowers
Sigh in the breeze and fill the air with sweet heady perfume
For they, they do not have square roots.

And this is the decimal point, the purpose of which
Is just vague and unknown, as you know. But we move it
Stupidly backwards and forwards. We call this mathematics.
And stupidly backwards and forwards
The drowsy-winged insects are flying in the meadows.
They call it lessons on maths.

They call it lessons on maths ; it is perfectly easy
If you have sense and use your tables, for the roots,
And the sines, and tans of angles, and, of course, your brains,
Which in our case we have not got, and the laburnum trees
Lift their yellow lamps, and dance ; and the black swifts dart rapidly downwards
For to-day we have lessons on maths.

Sonia Benson and Ella Whitfield, IV.D, Lancaster Girls' Grammar School.

1825. Sir, I have within the last hour seen more than, if not quite, 70 swans in my fields.—Earl of Lucan to the Lord Chamberlain, in a letter (8 December 1853) quoted (p. 132) by C. Woodham Smith in *The Reason Why* (1953). [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES

2505. *On magic squares.*

The following property of a certain type of magic square was not known to the authors and as it is capable of a simple proof it was thought that it might be of interest to readers of the *Gazette*.

" The inverse of the matrix consisting of the elements of a magic square, is itself a magic square, whose row and column sums are the reciprocals of those of the original square."

For, let $[a_{ij}]$ be the original magic square matrix, then

$$[a_{ij}]^{-1}=[A_{ji}/\Delta] \qquad \text{.......(1)}$$

where A_{ij} is the cofactor of a_{ij} in $|\,a_{ij}\,|$ and

$$\Delta=|\,a_{ij}\,|. \qquad \text{.......(2)}$$

Now
$$\sum_{i=1}^{n}\frac{\Delta_{ij}}{\Delta}=\frac{1}{\Delta}\sum_{i=1}^{n}A_{ij} \quad (j=1\,\ldots\,n) \qquad \text{.......(3)}$$

and $\sum\limits_{i=1}^{n}A_{ij}$ is the determinant formed by putting all a_{ij} (for fixed j) equal to unity in $|\,a_{ij}\,|$.

However, by adding all columns of $|\,a_{ij}\,|$ into the jth column, we obtain:

$$|\,a_{ij}\,|=\begin{vmatrix} a_{11}a_{12}\ldots a_{1j}\ldots a_{1n} \\ a_{21}\;\ldots\ldots\; a_{2j}\ldots a_{2n} \\ \\ \\ a_{n1}\;\ldots\ldots\; a_{nj}\ldots a_{nn} \end{vmatrix}=\begin{vmatrix} a_{11}\ldots \sum\limits_{j=1}^{n} a_{1j}\ldots a_{1n} \\ a_{21}\ldots \Sigma a_{2j} \quad \ldots\ldots \\ \\ \\ a_{n1}\ldots \Sigma a_{nj}\ldots a_{nn} \end{vmatrix}$$

but since $[a_{ij}]$ is a magic square :

$$\sum_{j=1}^{n}a_{ij}=C \quad (\text{for all } i),$$

whence

$$|\,a_{ij}\,|=C\begin{vmatrix} a_{11}a_{12}\ldots 1\ldots a_{1n} \\ a_{21}a_{22}\ldots 1\ldots a_{2n} \\ \\ \\ a_{n1}\;\ldots\ldots\; 1\ldots a_{nn} \end{vmatrix} \quad (\text{for all } j),$$

or, expanding in terms of the jth column and its co-factors,

$$|\,a_{ij}\,|=C\sum_{i=1}^{n}A_{ij} \quad (\text{for all } j), \qquad \text{.......(4)}$$

whence, from (3),

$$\sum_{i=1}^{n}\frac{A_{ij}}{\Delta}=\frac{1}{\Delta}\sum_{i=1}^{n}A_{ij}=\frac{|\,a_{ij}\,|}{C\Delta}=\frac{1}{C}\cdot \qquad \text{.......(5)}$$

In a similar manner, it may be shown that the row sums of the inverse matrix

are also equal to $1/C$, which proves the theorem.

The authors are indebted to Mr. R. Bird for suggesting this problem.

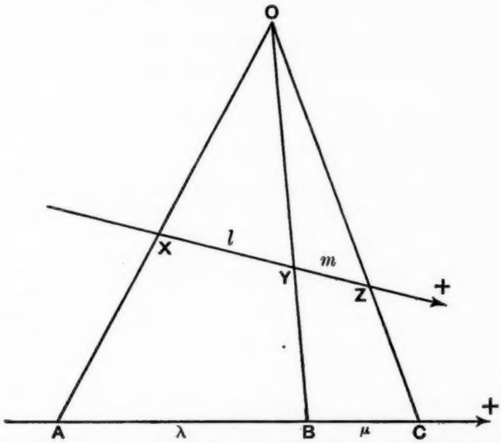
A. D. Booth, K. H. V. Booth

2506. *Note on a theorem from " Reciprocal Nomograms ", by C. V. Gregg.*

The theorem established by C. V. Gregg at the beginning of his article " Reciprocal Nomograms ", *Mathematical Gazette*, XXXVII (1953), p. 90, is amenable to a vectorial treatment which, as an example of the compact proofs vector methods sometimes provide, might be of interest.

For convenience introduce the following notation for the vectors appearing in Figure 1 of the above article (substantially reproduced here) :

$$|\mathbf{OX}|=x \qquad |\mathbf{OY}|=y \qquad |\mathbf{OZ}|=z$$
$$|\mathbf{OA}|=a \qquad |\mathbf{OB}|=b \qquad |\mathbf{OC}|=c.$$



The segments $AB=\lambda$, $BC=\mu$, $XY=l$, $YZ=m$, are positive when they are in the directions indicated as positive in the figure, otherwise negative. The equations of the theorem which are to be established then become, respectively :

$$\frac{\sin BOC}{x}+\frac{\sin COA}{y}+\frac{\sin AOB}{z}=0, \qquad \text{(i)}$$

$$\mu a/x-(\mu+\lambda)b/y+\lambda c/z=0. \qquad \text{(ii)}$$

To prove equation (i) note that

$$m\mathbf{OX}-(m+l)\mathbf{OY}+l\mathbf{OZ}=0. \qquad \text{(iii)}$$

The magnitudes of the vector products of (iii) with $\mathbf{OX}$ and $\mathbf{OZ}$ are, respectively,

$$lxz \sin AOC-(m+l)\,xy \sin AOB=0.$$

and

$$mzx \sin COA+(m-l)\,yz \sin COB=0.$$

These can be written as

$$(l/y)\sin COA+(m+l)\sin AOB/z=0,$$
$$(m/y)\sin COA+(m+l)\sin BOC/x=0.$$

Adding these two equations and factoring out $(m+l)$ yields equation (i).

Next, to prove equation (ii), write (iii) in the form

$$\mathbf{OY}=(m\mathbf{OX}+l\mathbf{OZ})/(m+l), \quad \text{.............................(iv)}$$

and similarly write

$$\mathbf{OB}=(\mu\mathbf{OA}+\lambda\mathbf{OC})/(\mu+\lambda). \quad \text{.............................(v)}$$

Since $\mathbf{OX}=x\mathbf{OA}/a, \quad \mathbf{OY}=y\mathbf{OB}/b, \quad \mathbf{OZ}=z\mathbf{OC}/c,$

equation (iv) becomes

$$y\mathbf{OB}/b=(mx\mathbf{OA}/a+lz\mathbf{OC}/c)/(m+l). \quad \text{.......................(vi)}$$

On equating corresponding coefficients in (v) and (vi) we get

$$ya\mu/bx(\mu+\lambda)=m/(m+l),$$
$$yc\lambda/bz(\mu+\lambda)=l/(m+l),$$

whence on adding we have

$$ya\mu/bx(\mu+\lambda)+yc\lambda/bz(\mu+\lambda)=1.$$

When this equation is rearranged, the equation (ii) results.

C. I. Lubin.

2507. *An elementary application of homography.*

Mr. Hopkins and Mr. Drazin * have embroidered a theorem on the triangle so tastefully that their readers may fail to see that it is the simplest of deductions from an obvious homography. Given a triangle ABC and a fixed point X, lines through B isoclinal for BX and through C isoclinal for CX are fixed lines r, q. If Z is a variable point of r, and if the line through A isoclinal for AZ cuts q in Y, then since the pencil $A(Y)$ is a reflection of the pencil $A(Z)$, the ranges (Y), (Z) are related homographically, and all the intersections of pairs of lines $Y'Z''$, $Z'Y''$ are collinear on the axis of the homography. To B as a Z corresponds C as a Y, and therefore in particular the intersection of every pair BY, CZ is on the axis. If r cuts AC in R and q cuts AB in Q, then to R as a Z corresponds Q as a Y, and therefore the intersection of BQ and CR is on the axis. If BX cuts q in V and CX cuts r in W, then since BV, BW are isoclinal at B and CV, CW are isoclinal at C, it follows that AV, AW are isoclinal at A, that is, that to W as a Z corresponds V as a Y and therefore the intersection of BV and CW also is on the axis. Thus the axis is AX.

E. H. N.

2508. *Approximate construction of the regular heptagon.*

There is a slightly more accurate approximate method of constructing a regular heptagon in a circle than that suggested in Note 2297 (XXXVI, No. 318). If we construct at the centre of the circle an angle $\tan^{-1}(5/4)$, which is $51° 20\frac{1}{2}'$, this gives a heptagon of side $2r \sin 25° 40\frac{1}{4}'$, that is, $0{\cdot}8664r$, as against Mr. Banner's $0{\cdot}8660r$. For a pentagon the angle $\tan^{-1} 3$, that is, $71° 34'$ is sufficiently accurate for young pupils. The great advantage of the tangent method is that it is general, and may be instructive, apart from foreshadowing the idea of tangent. I have used it in a recent series of arithmetic textbooks for primary pupils to connect numerical division with central symmetry. Thus, for $n \div 7$, the quotient is written as each corner of a regular heptagon inscribed in a circle, and the remainder at the centre.

R. S. Williamson.

* *Gazette*, XXXIV (1950), p. 129, Note 2144; XXXVII (1953), p. 55, Note 2328.

2509. *The four 4's problem.*

The problem of the four 4's—making as many consecutive integers as possible out of four 4's and the usual mathematical signs—has thrived on the parlour-game level for many years. The crucial question is : what signs are "usual"? If monstrosities such as . $\sqrt{(\cdot\dot{4})}$ $(=\frac{1}{15})$ are allowed (and they often are), there seems no reason why the ubiquitous logarithm sign should be excluded. If logarithms are admitted, any positive integer n can be expressed in terms of *three* 4's, since

$$\frac{\log\left\{\dfrac{\log 4}{\log(\sqrt{\sqrt{\sqrt{\ldots\sqrt{4}}}})}\right\}}{\log\sqrt{4}}=n,$$

where n is the number of square root signs in the series. Furthermore, any integer can also be expressed in terms of four x's, where x is any real number except 0 and 1, since

$$\frac{\log\left\{\dfrac{\log x}{\log(\sqrt{\sqrt{\sqrt{\ldots\sqrt{x}}}})}\right\}}{\log\left\{\dfrac{\log x}{\log\sqrt{x}}\right\}}=n,$$

where n is again the number of square root signs in the series. If $x=1$, it is only necessary to write $\cdot 1$ instead of 1 in the expression. Finally, any integer n can be expressed in terms of m x's, where x is any real number (other than 0 or 1) and m is any integer greater than 5, since

$$\frac{\log\left\{\dfrac{\log x}{\log(\sqrt{\sqrt{\sqrt{\ldots\sqrt{x}}}})}\right\}}{\log\left\{\dfrac{\log x}{\log\sqrt{x}}\right\}}-\frac{(x+x+\ldots+x)}{x}=n,$$

if there are $(m+n-5)$ square root signs in the series, and $(m-5)$ terms in the series of x's. This formula holds for $m=5$ if the first denominator is re-written as

$$\log\left(\frac{x+x}{x}\right)\cdot$$

D. G. King-Hele.

2510. *On the laws of friction.*

The following remarks are intended to draw attention to an inconsistency in the laws of static friction as they are stated in some widely used English textbooks. Something like the following proposition P is found.

P. *Limiting friction acts in the direction of the relative motion which would take place in the absence of friction.*

Granted the usual law of dynamic friction that the friction force acts in the direction of the relative motion, P means that the friction force is continuous in direction through the onset of motion. An example given below shows that P is not correct in general. Indeed P *happens* to be true in the great majority of textbook problems ; this probably accounts for its long survival.

A new approach must therefore be found to those problems which are usually solved by initial motion arguments. The problem of a rod on a rough table is solved below as an example.

The general conclusion may be drawn that all *purely statical* problems can be solved by using the single law that the total reaction must lie within the

cone of friction. Problems of *initial motion* should be determined by calculation of the *initial* accelerations by dynamical methods.

The counter-example to P is taken from Loney's *Statics*. p. 193, No. 3.

A uniform rod AB can turn freely about the fixed end A and rests with the other end B against a rough vertical wall, making an angle α with the wall. Shew that the end B may rest anywhere on an arc of a circle of angle

$$2\, tan^{-1}(\mu\, tan\, \alpha).$$

Let C be the foot of the perpendicular from A to the wall. If, following P, we assume the limiting frictional force to act in the direction *perpendicular to BC*, we obtain Loney's result immediately by taking moments about a vertical axis through A. This result is wrong. Acting on the rod are three forces which must all act in the vertical plane through the rod. Moreover, the total reaction at B must be within the cone of friction at B. Consequently in the limiting position the vertical plane through the rod will be tangent to the cone, touching it along a horizontal generator. Hence in the limiting case the frictional force is *horizontal*. This leads to the solution

$$2 \sin^{-1} (\mu \tan \alpha).$$

Problems Nos. 2 and 5 of the same series are similarly incorrect, as is problem 21 on page 128 of Routh's *Analytical Statics*, Vol. 1—a tripos question of 1885! Minchin in his *Statics*, Vol. 2, page 53, 3rd edition, solves essentially our counter-example by both methods, and asserts that the incorrect solution applies to a perfectly inelastic rod, and that the correct solution applies to an elastic, though very rigid, rod.

Consider now the following problem which is generally solved by using the instantaneous centre of the initial motion.

A straight uniform rod rests on a rough horizontal table (coefficient of friction μ) so that its weight, w per unit length, is supported uniformly and its ends are at the points $(0, 0)$ *and* $(l, 0)$. *Determine the greatest force R in the direction of the negative y-axis which can be applied to the rod at the point* $(0, 0)$ *without disturbing the equilibrium.*

Let the friction force per unit length have components $X(x)$, $Y(x)$ $(0 \leqslant x \leqslant l)$. The equations of equilibrium are

$$\int_0^l X(x)\, dx = 0, \quad \text{.................................(1)}$$

$$\int_0^l Y(x)\, dx = R \quad \text{.................................(2)}$$

$$\int_0^l xY(x)\, dx = 0, \quad \text{.................................(3)}$$

and the law of friction requires

$$X^2(x) + Y^2(x) \leqslant \mu^2 w^2 \quad (0 \leqslant x \leqslant l). \quad \text{........................(4)}$$

The problem therefore is to find two unknown functions X and Y, subject to the conditions (1), (3) and (4), to make R, given by (2), a maximum. Clearly we must take $X(x) = 0$ $(0 \leqslant x \leqslant l)$, because a non-zero X makes no contribution to R and strengthens the restriction (4) on Y.

We proceed to determine that function Y, subject to (3) and

$$|Y(x)| \leqslant \mu w \quad (0 \leqslant x \leqslant l). \quad \text{..........................(4')}$$

which makes R a maximum. Take any function Y_0 which satisfies (3) and (4′). A perturbation in Y_0 consisting of an increase for small values of x and for larger values of x a decrease, chosen so that (3) continues to hold, will increase

R. This advantageous process of perturbation can be continued until the usual solution

$$Y(x) = \mu w \quad (0 \leqslant x < l/\sqrt{2}),$$
$$Y(x) = -\mu w \quad (l/\sqrt{2} < x \leqslant l)$$

is obtained.

There remains the determination of the set of cases in which P is valid.

T. A. S. Jackson.

2511. *On Note* 1975 ; *Shortcutting in multiplication on a calculating machine.*

If we multiply a number by 9, using a hand-calculating machine, we may either make nine positive turns of the operating handle or, by considering 9 as $10-1$, one positive and one negative turn. This economy in the number of turns is called shortcutting : if a digit is obtained by successive additions we call it low, if by an addition followed by subtractions we call it high ; if to obtain the procedure which gives the lowest total number of turns a digit may be treated as high or low we call it neutral.

In Note 1975 Dr. S. Vajda gave a set of rules for determining whether digits should be high, neutral or low for most efficient shortcutting. His rules appear, however, to give the wrong answer in a few cases, since, for example, in 545 all digits are low, whereas his rules appear to allow 5 and the pair 45 to be both high. In the present note a somewhat simpler rule is given and proved, and the resulting saving in turns is then considered quantitatively.

We denote by $m(l)$ the sum of the digits in the number l : this is the number of turns necessary if no shortcutting is used in multiplying by l. We denote by $s(l)$ the number of turns used in multiplying by l when shortcutting is used most effectively.

We first need to prove a lemma which we state as

Theorem 1. If $\frac{5}{11}10^n < l < \frac{6}{11}10^n$, then $s(l) = s(10^n - l)$. If $0 \leqslant l < \frac{5}{11}10^n$, then $s(l) = s(10^n - l) - 1$.

Proof. This result can be easily verified for $n=1$; suppose it is true for $n=N$. To establish it for $n=N+1$ it is sufficient to consider the cases $0 < l < 5 \,.\, 10^N$ since the result can be verified immediately for $l = 5 \,.\, 10^N$ and for $5 \,.\, 10^N < l \leqslant 10^{N+1}$ we can consider $10^{N+1} - l$ instead. Let $l = \lambda \,.\, 10^N + \mu$ where $0 \leqslant \lambda \leqslant 4$, $0 \leqslant \mu \leqslant 10^N$. Then

$$s(l) = \min\{\lambda + s(\mu),\ \lambda + 1 + s(10^N - \mu),\ 10 - \lambda + s(10^N - \mu),\ 11 - \lambda + s(\mu)\}.$$

Since $|\, s(10^N - \mu) - s(\mu) \,| \leqslant 1$ by hypothesis, and $\lambda \leqslant 4$, we have $s(l) = \lambda + s(\mu)$.

$$\begin{aligned}\text{Also}\quad s(10^{N+1} - l) &= \min\{9 - \lambda + s(10^N - \mu),\ 10 - \lambda + s(\mu),\ \lambda + 1 + s(\mu),\\ &\qquad\qquad \lambda + 2 + s(10^N - \mu)\}\\ &= \min\{9 - \lambda + s(10^N - \mu),\quad \lambda + 1 + s(\mu)\}.\end{aligned}$$

For $\lambda = 0$, 1, 2 or 3, $s(10^{N+1} - l) = \lambda + 1 + s(\mu) = s(l) + 1$.

For $\lambda = 4$, $s(10^{N+1} - l) = \lambda + 1 + s(\mu) = s(l) + 1$ if and only if $s(10^N - \mu) \geqslant s(\mu)$, *i.e.* if and only if $\mu < \frac{6}{11}10^N$, so that

$$l < 4 \,.\, 10^N + \tfrac{6}{11} \,.\, 10^N = \tfrac{5}{11} \,.\, 10^{N+1}.$$

But if $l > \frac{5}{11}10^{N+1}$, then $\mu > \frac{6}{11}10^N$ and $s(10^N - \mu) = s(\mu) - 1$, so that

$$s(10^{N+1} - l) = 4 + s(\mu) = s(l).$$

This proves the theorem in the case $n = N+1$ and hence, by induction, for all cases.

We can now give the rule for shortcutting which we state as

Theorem 2. For the number $\lambda \,.\, 10^n + \mu$ where $0 \leqslant \lambda \leqslant 9$, $0 < \mu \leqslant 10^n$, λ is low if it is 0, 1, 2, 3 or 4 and high if it is 6, 7, 8 or 9. If $\lambda = 5$ it is low if $\mu < \frac{5}{11}10^n$, neutral if

$$\tfrac{5}{11}10^n < \mu < \tfrac{6}{11}10^n,$$

and high if $\frac{6}{11}10^n < \mu$.

Proof.

$$s(l) = \min\{\lambda + s(\mu),\ \lambda + 1 + s(10^n - \mu),\ 10 - \lambda + s(10^n - \mu),\ 11 - \lambda + s(\mu)\},$$

the first two terms applying when λ is low and the last two when λ is high. For $\lambda = 0, 1, 2, 3, 4$ the first two terms are smaller than the others, so that λ is low; for $\lambda = 6, 7, 8, 9$ they are larger.

For $\lambda = 5$, $s(l) = \min\{5 + s(\mu), 5 + s(10^n - \mu)\}$.

5 is low if $5 + s(\mu) < 5 + s(10^n - \mu)$, which by Theorem 1 occurs if and only if $\mu < \frac{5}{11}10^n$. The other two cases follow similarly.

As an example consider $l = 6447$. 6 is high, so that we take the number to be $10000 - 3553$. 3 is low, so that $3553 = 3000 + 553$. The first 5 is neutral, giving either $500 + 50 + 3$ or $1000 - 447$. Finally 447 must be $400 + 40 + 10 - 3$. Thus the two possibilities are $10000 - 3000 - 5000 - 50 - 3$ or

$$10000 - 4000 + 400 + 50 - 3,$$

both giving $s(6447) = 17$.

To discuss quantitatively the saving in work due to shortcutting we shall need to know the values of

$$M(n) = \sum_{1}^{10^n - 1} m(l) \text{ and } S(n) = \sum_{1}^{10^n - 1} s(l).$$

Theorem 3. $M(n) = \frac{9}{2}n \,.\, 10^n$.

Proof. Consider all numbers less than 10^n; by grouping together those with the same nth digit (counting from the right) we have

$$M(n) = 10M(n-1) + (0 + 1 + \ldots + 9)10^{n-1} = 10M(n-1) + 45 \,.\, 10^{n-1}.$$

Since $M(1) = 45$ this establishes the result by induction.

Theorem 4. $S(n) = (\frac{27}{11}n + \frac{60}{121})10^n - \frac{1}{242}(121 + (-1)^{n-1})$.

Proof. Consider blocks of 10^{n-1} numbers with nth digits 0, 1, 2, 3, 4. These give

$$\sum_{1}^{5.10^{n-1}-1} s(l) = 10^{n-1}(0 + 1 + 2 + 3 + 4) + 5S(n-1) = 10^n + 5S(n-1).$$

From Theorem 1, $s(10^n - l) = s(l)$ for $5 \,.\, 10^{n-1} < l < 10^n$, except in $[\frac{5}{11}10^n]$ cases when $s(10^n - l) = s(l) - 1$.

Since finally $s(5 \,.\, 10^{n-1}) = 5$, we have

$$S(n) = 5 + 2(10^n + 5S(n-1)) + [\tfrac{5}{11} \,.\, 10^n].$$

Now $[\frac{5}{11}10^n] = \frac{5}{11}10^n - \frac{1}{2} + \frac{1}{22}(-1)^n$, so that

$$S(n) = 10S(n-1) + \tfrac{27}{11}10^n + \tfrac{9}{2} + \tfrac{1}{22}(-1)^n.$$

By direct calculation $S(1) = 29$, and this serves to establish the result by induction.

We shall denote by $w(n)$ the ratio of the average work done in multiplying, using shortcutting with a machine whose multiplier register has n places, to the average amount of work done without shortcutting. If all numbers from 0 to $10^n - 1$ are equally likely, then $w(n)$ is the sum of the number of turns used, when shortcutting is employed, for all numbers from 0 to $10^n - 1$,

divided by $M(n)$. Now it is not possible to shortcut in the nth place; consequently the sum of the number of turns used is

$$10S(n-1)+(0+1+\dots+9)\,10^{n-1},$$

obtained by considering blocks of numbers with the same nth digit.

Hence $w(n)=\{45\,.\,10^{n-1}+10S(n-1)\}/M(n)$, which gives the following values (rounded off):

n	1	2	3	4	5	6	7	8	9	10	11
$w(n)$	1	·82	·73	·69	·66	·64	·63	·62	·61	·602	·597

The limiting value of $w(n)$ as n tends to infinity is $6/11$, representing a saving of 45·45%.

Some objections may be raised to the practical validity of the assumptions made. In the first place, since shift of the carriage of the machine takes time, we should perhaps count a shift of the carriage through one place as equivalent to p turns. This will alter the rule for shortcutting only for the first significant figure—the effect being to increase the critical values 5/11 and 6/11. Unless p is an integer no neutral numbers occur. The efficiency of shortcutting will drop, partly because it is less often applicable and partly because we have to do an amount of work in carriage shifting which is the same whether we shortcut or not. The limiting value of $w(n)$ changes from $6/11$ to $(\frac{27}{11}+p)/(\frac{9}{2}+p)$.

In the second place we may allow for the inconvenience of changing the direction of motion of the operating handle—that is, changing from addition to subtraction and *vice versa*—by counting each change of direction as equivalent to q turns. The result of this is to replace the results of Theorems 1, 2 and 4 by rather complicated ones, but we may somewhat overestimate the effect by counting the number of changes of direction that take place with the ordinary shortcutting process. The total number of changes of direction that take place among the numbers 0 to 10^n-1 can be shown to be

$$\left(\tfrac{441}{1100}n-\tfrac{21}{12100}\right)10^n+\tfrac{1}{121}(-1)^{n-1},$$

so that the limiting value of $w(n)$ is less than $(\frac{27}{11}+\frac{441}{1100}q)/\frac{9}{2}$. For $q=1$ this is $\frac{349}{550}=63{\cdot}45\%$.

In the third place we may reject the notion that all numbers are equally likely; if the probability of a number being used as a multiplier is proportional to its reciprocal, then

$$w(n)=\left(\sum_{1}^{10^n-1}\frac{s(r)}{r}\right)\Bigg/\left(\sum_{1}^{10^n-1}\frac{m(r)}{r}\right).$$

Since this hypothesis attaches more weight to the lower numbers, for which shortcutting is not used, $w(n)$ will be increased. The series arising do not seem amenable to summation in simple terms; by direct computation we have in this case (with shortcutting in all places) $w(1)=\cdot778$, $w(2)=\cdot714$. This suggests that the limiting value may be about ·65 as compared with ·55 in the case of equal probabilities.

H. J. GODWIN.

2512. *Cubic and quartic equations.*

1. Cardan's solution of a cubic follows naturally and easily from the identity

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z), \quad\dots\dots\dots\dots(1)$$

which shows that $\qquad x^3-3yzx+y^3+z^3=0$

if $\qquad x=-y-z,\ -\omega y-\omega^2 z,\ \text{or}\ -\omega^2 y-\omega z. \quad\dots\dots\dots\dots(2)$

We accordingly give y and z any values satisfying

$$-3yz = a, \quad y^3 + z^3 = b,$$

and (2) is then the solution of

$$x^3 + ax + b = 0.$$

2. The fourth-degree analogue of (1), namely

$$\begin{vmatrix} x & y & z & t \\ t & x & y & z \\ z & t & x & y \\ y & z & t & x \end{vmatrix} = \begin{array}{l} (x+\ y+z+\ t) \\ \times(x-\ y+z-\ t) \\ \times(x+iy-z-it) \\ \times(x-iy-z+it), \end{array} \qquad \text{.....(3)}$$

leads likewise to the solution of a quartic. The expansion of the determinant is

$$x^4 - 2(2yt + z^2)x^2 + 4z(y^2 + t^2)x + (2yt - z^2)^2 - (y^2 + t^2)^2,$$

so if y, z, t have any values satisfying

$$\begin{aligned} -2(2yt + z^2) &= a, \\ 4z(y^2 + t^2) &= b, \\ (2yt - z^2)^2 - (y^2 + t^2)^2 &= c, \end{aligned} \qquad \text{.....(4)}$$

then the solution of $x^4 + ax^2 + bx + c = 0$ is, from (3),

$$x = -z \pm (y + t), \; z \pm i(y - t).$$

To find the roots we use the relations

$$2yt = -a/2 - z^2, \quad y^2 + t^2 = b/(4z) \qquad \text{.....(5)}$$

to eliminate y and t from (4). This gives the resolvent

$$4z^6 + 2az^4 + (a^2/4 - c)z^2 - b^2/16 = 0,$$

from which a real or purely imaginary value of z can be found. We then find $y + t$ and $y - t$ from (5).

3. Identities like (1) and (3) exist for all degrees. So the method described for cubics and quartics would be quite general, if only. . . .

H. Lindgren.

2513. *On Note* 2339.

The exact formula for the shaded area is

$$A = \frac{1}{2}\left(\frac{c^2 + h^2}{2h}\right)^2 \left[\left(\frac{2ch}{c^2 - h^2}\right) - \frac{1}{3}\left(\frac{2ch}{c^2 - h^2}\right)^3 + \frac{1}{5}\left(\frac{2ch}{c^2 - h^2}\right)^5 - \dots\right] - \frac{c}{2}\left(\frac{c^2 - h^2}{2h}\right) \qquad \text{......(1)}$$

provided $\qquad 2ch/(c^2 - h^2) \leqslant 1.$

If c is fixed and h is small, and if we neglect throughout powers of h higher than the second, we obtain the approximation $A = \frac{2}{3}hc$ (with an error $O(h^3)$).

It is quite clearly inadmissible to use only the first term of the infinite series in (1) since a term linear in h arises from multiplying $\frac{1}{3}\left(\frac{2ch}{c^2 - h^2}\right)^3$ by

$$\frac{1}{2}\left(\frac{c^2 + h^2}{2h}\right)^2.$$

Hazel Perfect.

2514. *On Notes* 2309 *and* 2117

Integral solutions of

$$r^2 + r(x+y) = xy$$

are given by

$$r : x : y = n(m-n) : m(m-n) : n(m+n).$$

C. V. GREGG.

2515. *A note on squares.*

If n be a positive or negative integer the necessary and sufficient condition that

(1) $5n^2+4$ is a square is that $n = \pm\dfrac{(1+\sqrt{5})^{2r} - (1-\sqrt{5})^{2r}}{2^{2r} \cdot \sqrt{5}}$ $(r=1, 2, \ldots)$.

(2) $5n^2-4$ is a square is that $n = \pm\dfrac{(1+\sqrt{5})^{2r+1} - (1-\sqrt{5})^{2r+1}}{2^{2r+1} \cdot \sqrt{5}}$;

(3) $5n^2+4n$ is a square is that $n = \dfrac{\{(1+\sqrt{5})^r - (1-\sqrt{5})^r\}^2}{5 \cdot (-4)^r}$.

C. V. GREGG.

2516. *Cyclic permutation of digits.*

The method for testing whether a given number is divisible by p or not, where p is any factor of $(10^r - 1)$ can be had from Note 1920 (*Math. Gazette*, July, 1946). In this Note a special property of numbers having nr digits which are divisible by p is given:

If a number N, having nr digits, is divisible by p, where p is any factor of (10^r-1), then any number with the same digits cyclically permuted will also be divisible by p.

We shall show that the result holds for one cyclic permutation. Let

$$N = a_1 \cdot 10^{nr-1} + a_2 \cdot 10^{nr-2} + \ldots + a_{nr-1} \cdot 10 + a_{nr},$$

$$N_1 = a_2 \cdot 10^{nr-1} + a_3 \cdot 10^{nr-2} + \ldots + a_{nr} \cdot 10 + a_1.$$

Then
$$10N - N_1 = a_1(10^{nr} - 1);$$

thus $10N - N_1$ is divisible by $(10^r - 1)$ and hence the result.

S. PARAMESWARAN.

2517. *On Note* 2285 *: multiple exponentials.*

Let a be a positive real number less than unity, and define a_n for all positive integers n by induction thus:

$$a_0 = 1, \quad a_n = a^{a_{n-1}} \quad (\text{so that } a_1 = a).$$

For typographical convenience, we shall frequently denote "a to the power b to the power c ..." by $\{a, b, c, \ldots\}$.

Then the sequence a_n is convergent if $e^{-e} \leqslant a < 1$, but oscillates if $a < e^{-e}$. For,

$$a_0 = 1, \quad a_0 > a_1;$$
$$a_2 = a^a < 1^a = a_0, \quad a_2 = a^a > a^1 = a_1;$$
$$a_3 = \{a, a_2\} > \{a, a_0\} = a_1, \quad a_3 = \{a, a_2\} < \{a, a_1\} = a_2;$$

and if
$$a_{2n-2} > a_{2n}, \quad \{a, a_{2n-2}\} < \{a, a_{2n}\},$$

and so
$$a_{2n} = \{a, a, a_{2n-2}\} > \{a, a, a_{2n}\} = a_{2n+2}.$$

Thus by induction a_{2n} is monotonic decreasing, and since each term is positive

we have

$$\lim a_{2n} = U \geqslant 0.$$

Similarly by induction a_{2n-1} is monotonic increasing, and each term is less than unity, and hence

$$\lim a_{2n-1} = V \leqslant 1.$$

Further, by induction, $a_{2n-1} < a_{2n}$, and so $V \leqslant U$. Moreover, $a_{2n} < a_0 = 1$, and so $U \leqslant 1$.

Now,

$$U = \lim a_{2n+2} = \lim \{a, a, a_{2n}\} = \{a, a, U\},$$

and similarly

$$V = \{a, a, V\}.$$

Thus U and V both satisfy the equation

$$\phi(x) \equiv x - \{a, a, x\} = 0.$$

If x is a root of this equation, then

$$\log \log (1/a) + x \log a - \log \log (1/x) = 0,$$

and if we write $\psi(x)$ for the left-hand side of this last equation, $\psi(x)$ vanishes with, and has the same sign as, $\phi(x)$. Then

$$\psi'(x) = -1/x \log x + \log a,$$

and

$$\psi'(x) = 0 \text{ when } x^x = e^{1/\log a}.$$

Now the equation $x^x = c$ has no real roots for x if $c < c_0 = \{e, -1/e\}$, one root if $c = c_0$, and two real roots if $c_0 < c < 1$, since x^x has a minimum value c_0 at $x = 1/e$. Thus $\psi'(x) = 0$ has no root, one root, or two roots according as a is greater than, equal to, or less than e^{-e}.

As $x \to 0$, $\psi(x) \to -\infty$, and as $x \to 1$, $\psi(x) \to +\infty$, so $\psi(x) = 0$ has an odd number of roots in (0, 1). Hence if $a \geqslant e^{-e}$, the equation $\psi(x) = 0$ has only one root in (0, 1) and hence so has $\phi(x) = 0$. *Thus in this case* $U = V$, *and the sequence* a_n *converges.*

Suppose now $a < e^{-e}$, so that the equation

$$x^x = b = e^{1/\log a}$$

has two roots x_1, x_2 in (0, 1), where $x_1 < 1/e < x_2$. We have

$$x \lesseqgtr x_1, \quad x^x \gtreqless b, \quad \psi'(x) \gtreqless 0,$$

so that $x = x_1$ is a maximum of $\psi(x)$; similarly $x = x_2$ is a minimum of $\psi(x)$. Using $\psi'(x_1) = 0$, we have

$$\begin{aligned}\psi(x_1) &= \log \log (1/a) - \log \log (1/x_1) + x_1 \log a \\ &= [\log (1/x_1) - \log \log (1/x_1)] - \log \log (1/x_1) - 1/\log (1/x_1) \\ &= r - (1/r) - 2 \log r = \lambda(r),\end{aligned}$$

where

$$r = \log (1/x_1) > \log e = 1.$$

Then $\lambda'(r) = (1-r)^2/r^2 > 0$ for $r > 1$, and so

$$\lambda(r) > \lambda(1) = 0, \quad \psi(x_1) > 0.$$

Further, $\psi(1/e) = \log \log (1/a) - e^{-1} \log (1/a) < 0$, since $(\log z)/z < 1/e$ for $z > e$. Thus $\psi(1/e) < 0$, and hence the equation $\psi(x) = 0$ and so, the equation $\phi(x) = 0$, has three roots β_1, β_2, β_3 in (0, 1), where

$$0 < \beta_1 < x_1 < \beta_2 < 1/e < \beta_3 < 1.$$

Finally, always supposing that $a < e^{-e}$, $U = \beta_3$, and $V = \beta_1$. For, $a_0 = 1 > 1/e$, and if $a_{2n-2} > 1/e$, then

$$a_{2n} = \{a, a, a_{2n-2}\} > \{a, a, 1/e\} > 1/e,$$

since $\psi(1/e) < 0$ and so $\phi(1/e) < 0$. Thus by induction $a_{2n} > 1/e$ and so

$$\lim a_{2n} = U \geqslant 1/e, \quad \text{or } U = \beta_3.$$

If $a_{2n-1} < x_1$, then

$$a_{2n+1} = \{a, a, a_{2n-1}\} < \{a, a, x_1\} < x_1,$$

since $\psi(x_1) > 0$ and so $\phi(x_1) > 0$. To complete the induction, we must show that $a_1 < x_1$. Now $z \geqslant e \log z$ for all positive z, and so

$$\frac{1}{\sqrt{x_1}} \geqslant e \log \frac{1}{\sqrt{x_1}} > 2 \log \frac{1}{\sqrt{x_1}} = \log \frac{1}{x_1},$$

that is,

$$\frac{1}{x_1} > \left(\log \frac{1}{x_1}\right)^2,$$

$$1/x_1 \log (1/x_1) > \log (1/x_1),$$

or

$$\log a = 1/x_1 \log x_1 < \log x_1,$$

so that

$$a < x_1.$$

Hence by induction $a_{2n-1} < x_1$, and thus

$$\lim a_{2n-1} = V \leqslant x_1, \text{ or } V = \beta_1.$$

Thus in this case, $U \neq V$ and the sequence a_n oscillates with β_3, β_1 as upper and lower limits.

A. G. Vosper

2518. *A note on a Simson line property.*

It is well known that if H' is a point on the circumcircle of a triangle ABC with orthocentre H then the Simson line of H' is also the Simson line of a point A' on the circumcircle of BCH with respect to this triangle. It is likewise the Simson line of points B', C' on the circumcircles of the triangles CAH, BAH with respect to these triangles. Furthermore the points $A'B'C'H'$ form an orthocentric set congruent and antihomothetic to the set $ABCH$. These properties follow quite easily from the result that the Simson line of H' bisects HH' at a point on the nine-point circle. The antihomothetic centre of the two systems is the point at which their two nine-point circles touch.

This means that the four points A, B, C, H are on the circumcircles of the four points A', B', C', H' and that all four Simson lines are once again the same line as before. It is the Simson line of eight systems simultaneously.

If we imagine the Simson line to remain fixed and the two orthocentric sets to move we obtain an interesting motion. The four points of each set move round the circumcircles of the other set while the two nine-point circles slide round one another. In each system the envelope of the Simson line is a three-cusped hypocycloid. These curves slide on the fixed line always cutting off a fixed segment on it because of the constant tangent property of the curve. We have made an animated cartoon film of this motion which enables all these properties to be seen.

Other interesting properties may be derived quite easily. Two further tangents may be drawn from the antihomothetic centre to the hypocycloids. These are perpendicular and rotate at uniform speed. The path of the cusps of the curves is a nephroid; and it is an interesting problem to determine the space and body centrodes of the motion.

T. J. Fletcher.

2519. *Concerning* $t + \surd(t^2 - 1)$.

1. If t is real and greater than 1, we can easily see that

$$2\{t + \surd(t^2 - 1)\} = \{\surd(t+1) + \surd(t-1)\}^2,$$

the positive root being taken in each case. A number of students now proceed as follows when discussing values of t less than -1. On putting $t = -x$, the above identity becomes

$$\begin{aligned} 2\{-x + \surd(x^2 - 1)\} &= \{\surd(-x+1) + \surd(-x-1)\}^2 \\ &= \{i\surd(x-1) + i\surd(x+1)\}^2 \\ &= i^2\{\surd(x-1) + \surd(x+1)\}^2 \\ &= -2\{x + \surd(x^2 - 1)\}. \end{aligned}$$

The fallacy is obvious, and this type of argument should be discouraged.

2. The expression occurs in fluid mechanics as a curve factor for transforming the shaded area of the figure in the z-plane into the upper half of the t-plane. In particular, we consider the case where the sections AB, CD, EF are straight, with AB perpendicular to CD and EF.

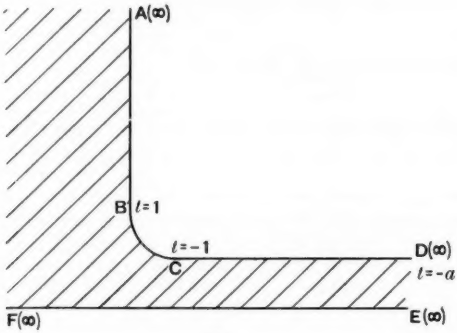


Fig.

At first sight the equations

$$A\frac{dz}{dt} = \frac{\surd(t+1) + \surd(t-1)}{(t+a)} \qquad \text{.............................(i)}$$

and

$$A\frac{dz}{dt} = \frac{\surd 2 \,.\, \{t + \surd(t^2 - 1)\}^{1/2}}{(t+a)} \qquad \text{...........................(ii)}$$

appear identical, and so either would seem to be the required transformation. However, on putting $t = -\alpha$, $\alpha > 1$, and taking the principal values of the square roots in (i), we soon notice the difference (compare with § 1 above). As a matter of fact the expression

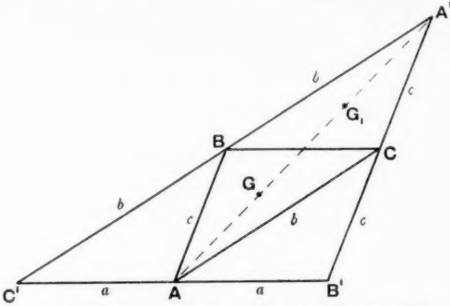
$$\{\alpha - \surd(\alpha^2 - 1)\}^{1/2}/(a - \alpha)$$

can at first decrease and later increase to infinity as α goes from 1 to a. Thus (ii) as it stands will not be valid for irrotational fluid motion. Equation (i) will be valid provided we always take the principal values of the square roots.

G. Power.

2520. *Moment of inertia of a triangular lamina.*

Let the moment of inertia of a uniform triangle about its centre of gravity be Mk^2, M being its mass and k^2 a function of its sides a, b and c.



Then the moment of inertia of the triangle $A'B'C'$ is $16Mk^2$, and this is the sum of the moments of inertia about G of its four parts, so that

$$16Mk^2 = Mk^2 + \Sigma(Mk^2 + M\,.\,GG_1^{\,2}).$$

Hence

$$\begin{aligned} 12k^2 &= \Sigma GG_1^{\,2} \\ &= \tfrac{1}{9}\Sigma A'A^2 \\ &= \tfrac{1}{9}\Sigma(2b^2 + 2c^2 - a^2) \\ &= \tfrac{1}{3}\Sigma a^2. \end{aligned}$$

Thus the moment of inertia about its centre of gravity of a uniform triangular lamina of mass M with sides a, b and c is

$$M(a^2 + b^2 + c^2)/36.$$

W. Hope-Jones.

2551. *Numerical solution of equations.*

The following application of infinite geometric series to a well-known iterative procedure for the solution of equations (described for certain algebraic equations in the *Gazette*, No. 165, p. 339) cannot be new, but I have no recollection of having seen it in print. Calculus is not necessarily required, but the process can be further speeded up by noting that the common ratio of the series may be expressed as the quotient of two differential coefficients.

The ideas may be exemplified by treating the example given by E. H. Bateman in the *Gazette*, No. 320, p. 100, namely

$$f(x) \equiv x^3 - 4x^2 + 5 = 0.$$

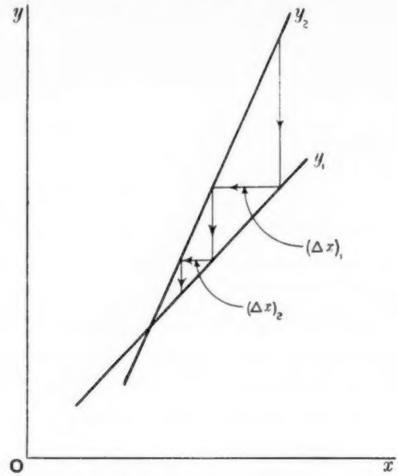
The solution of the equation is considered as the determination of the abscissa of the intersection of $y_1 = x^3$, $y_2 = 4x^2 - 5$. Note that this type of replacement is not unique.

Since $\quad f(1) = 2$ and $f(2) = -3$, $\quad x \simeq 1{\cdot}4$;

and then $\quad f(x) = -0{\cdot}096$, $\quad y_1 = 2{\cdot}744$, $\quad y_2 = 2{\cdot}84$.

The diagram is intended to show the neighbourhood of the intersection, the curves being replaced by straight lines. The convergent iterative process

indicated on the diagram is used, starting from $x = 1\cdot4$, $y = 2\cdot744$: x, y being calculated in succession from $y = x^3$, $x = \frac{1}{2}\sqrt{(y+5)}$.



Successively, retaining four decimal places,

$$\begin{array}{ccc} x & & y \\ 1\cdot4 & \longrightarrow & 2\cdot744 \\ & \swarrow & \\ 1\cdot3914 & \longrightarrow & 2\cdot6937 \\ & \swarrow & \\ 1\cdot3869 & & \end{array}$$

so that $(\Delta x)_1 = 0\cdot0086$, $(\Delta x)_2 = 0\cdot0045$.

Noting that the successive Δx are, approximately, in geometric progression, it is clear that a better approximation to the abscissa of the point of intersection is given by

$$1\cdot4 - \frac{0\cdot0086}{1 - \dfrac{45}{86}} = 1\cdot4 - 0\cdot0180 = 1\cdot3820,$$

which is correct to four decimal places.

Starting the iteration from the new value of x, and retaining seven decimal places, in succession

$$\begin{array}{ccc} x & & y \\ 1\cdot3820 & \longrightarrow & 2\cdot6395150 \\ & \swarrow & \\ 1\cdot3819836 & \longrightarrow & 2\cdot6394209 \\ & \swarrow & \\ 1\cdot3819751 & & \end{array}$$

and the new approximation is

$$x = 1\cdot3820 - \frac{0\cdot0000164}{1 - \frac{85}{164}}$$

$$= 1\cdot3820 - 0\cdot0000340$$
$$= 1\cdot3819660$$

agreeing with E. H. Bateman's value.

The process can be speeded up by noting that the common ratio of the geometrical progression,

$$\frac{(\Delta x)_2}{(\Delta x)_1} = \frac{dy_1}{dx} \Big/ \frac{dy_2}{dx} \quad \text{(in absolute value)},$$

so that only $(\Delta x)_1$ need be computed.

S. J. Tupper.

2522. *The ladder problem.*

1. The ladder problem is probably due for another wave of popularity. I take the case in which the two ladders are 20 ft. and 30 ft. long, and their meeting point is 10 ft. above the ground; and I work in 10 ft. units. The solution of a cumbersome quartic equation is to be avoided.

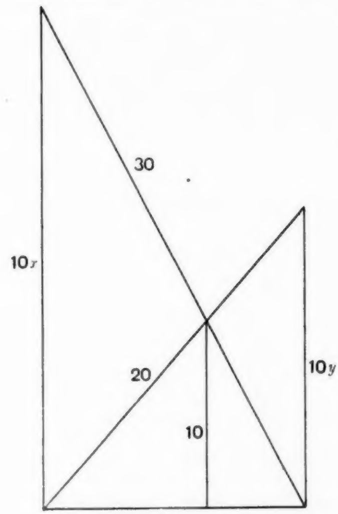


Fig.

2. Let x and y be the heights of the tops of the longer and shorter ladders from the ground, in 10 ft. units. Then

$$x^2 - y^2 = 3^2 - 2^2 = 5, \quad \text{.................................(i)}$$

and $$1/x + 1/y = 1. \quad \text{.................................(ii)}$$

Equation (i) is satisfied by $x = \surd 5 \,.\, \sec\theta$, $y = \surd 5 \,.\, \tan\theta$. Substituting these

values in (ii), we have

$$\cos\theta + \cot\theta = \sqrt{5},$$

which is very simply solved by any approximative method.

3. If we satisfy equation (i) by taking

$$x = \frac{\sqrt{5}}{2}\left(\frac{1}{u} + u\right), \quad y = \frac{\sqrt{5}}{2}\left(\frac{1}{u} - u\right),$$

then (ii) becomes

$$u^4 - \frac{4}{\sqrt{5}}u - 1 = 0, \quad \text{(iii)}$$

which is a quartic *in standard form*; and if we assume

$$(u^2 + ku + m)(u^2 - ku + n) = 0,$$

we arrive at

$$k^6 + 4k^2 = 3{\cdot}20,$$

which is a cubic for k^2 *in standard form.*

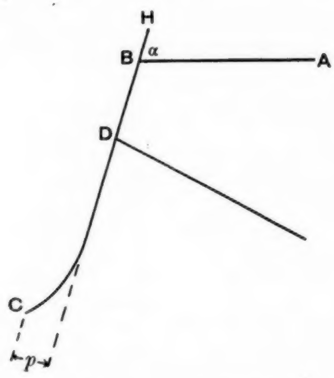
If we use hyperbolic functions for the substitution, then the standard form does not arise unless the equation is solved for $\exp(-u)$ and not $\exp(+u)$; hence the above arrangement.

G. A. Clarkson.

2523. *Bicycle frame. Front fork design.*

Nowadays, on horizontal ground, the tube AB is horizontal. The fork CD is bent as shown, the distance p being known as offset.

One condition for light steering is that turning of the handlebars should cause no appreciable vertical movement of B. A first approximation to the structural requirement for this condition may be found by considering turning through a right angle.



In this turn, if ABD remains in a vertical plane, the vertical component between C and B is increased by $p \, . \cos\alpha$. At the same time the front wheel comes into a plane inclined to the horizontal at angle α, so that C is lower by $R(1 - \sin\alpha)$, R being the wheel radius.

The condition requires that

$$p \, . \cos\alpha = R(1 - \sin\alpha).$$

i.e. $$p = R \,.\, \tan \tfrac{1}{2}\phi, \quad \text{where } \phi = 90^\circ - \alpha.$$

For a 26″ wheel with a head angle $\alpha = 70^\circ$ this gives $p = 2{\cdot}28''$.

Referring to horizontal and vertical axes through C, the equation of the front fork column is

$$x \cos \phi - y \sin \phi = R \tan \tfrac{1}{2}\phi.$$

This cuts the ground, where $y = -R$, at $x = -R \tan \frac{1}{2}\phi$. That is, the front column, if extended, strikes the ground in advance of the point of contact of the front wheel by a distance equal to the offset. With $\alpha = 70^\circ$ the same line just about bisects the vertical from C to the ground.

More generally, a turn of x° can be shown to produce a vertical displacement of B of an amount

$$\begin{aligned} & R\{1 - \sqrt{(1 - \sin^2 \phi \sin^2 x)}\} - p \,.\, \sin \phi \,.\, (1 - \cos x) \\ & \simeq \tfrac{1}{2}R \,.\, \sin^2 \phi \,.\, \sin^2 x - p \,.\, \sin \phi (1 - \cos x) \\ & = 2R \,.\, \sin^2 \tfrac{1}{2}\phi (\cos^2 \tfrac{1}{2}\phi \,.\, \sin^2 x - 2 \sin^2 \tfrac{1}{2}x), \quad \text{with } p = R \tan \tfrac{1}{2}\phi. \\ & = 2R \,.\, 2 \sin^2 \tfrac{1}{2}\phi \,.\, \sin^2 \tfrac{1}{2}x (2 \cos^2 \tfrac{1}{2}\phi \,.\, \cos^2 \tfrac{1}{2}x - 1). \end{aligned}$$

For any given value of ϕ, this lift reaches a maximum when $2 \cos^2 \frac{1}{2}\phi \,.\, \cos x = 1$.

With a head angle of 70° the maximum occurs at $x = 59^\circ$.

The following table shows how small is the vertical displacement (V) for the same machine.

x°	5	10	20	30	45	60	75
V''	·002	·008	·038	·060	·109	·129	·093

(The above investigation was suggested by an article in *The Bicycle* of 15 October, 1952.)

W. More.

2524. *The equivalent simple cranium.*

During the War I watched a tailor fitting A.T.C. recruits with caps. He measured each head with a tape and with little hesitation called out the appropriate size to his assistant. I asked him how it was done. "Well, sir," he said, "I don't really know, but a 22-inch head takes a size 7 hat, and for every half-inch more or less I allow ⅛ inch in the size, and it seems to work ".

A. P. R.

2525. *π in the Stillroom.*

If spheres are packed on a simple cubic system, in which their relative positions are those of the eggs in a crate of eggs, the material of the spheres occupies $\pi/6$ or 0·52 of the total space, and 0·48 of the space is empty. With the closest possible packing the spheres occupy $\pi\sqrt{2}/6$ of the space and only 0·26 is empty. In the preservation of almost-spherical fruit (currants, gooseberries, plums, etc.) failure to achieve close-packing leads to a waste of sugar and to an inefficient use of the jars. For example, a so-called 3-pound size of Kilner jar holds 40 ounces of water. If the jar is packed with dry fruit the amount of syrup needed to fill the jar may vary from 19 ounces to 10 ounces (fluid), and as a pint of the syrup normally used contains half-a-pound of sugar the waste may be ¼ lb. per jar. And seven jars might be needed when five should suffice.

The makers of water-glass for the preservation of eggs suggest that one-third of the containing vessel should be filled with liquid, but eggs are far from spherical, and it would be unwise to shake violently to encourage close packing.

A. P. R.

2526. *The midpoints of the diagonals of a complete quadrilateral.*

If L, M, N are the midpoints of the diagonals AC, BD, FG; then L, M, N are collinear. The following proof by projection can hardly be new, but it is not in any text-book I know.

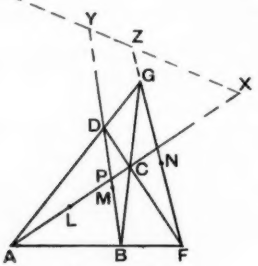


FIG. 1.

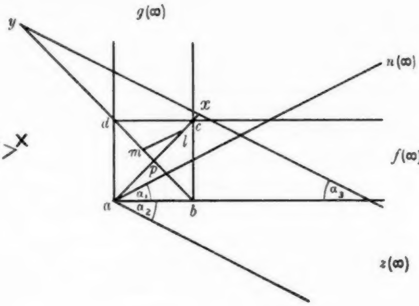


FIG. 2.

Let X, Y, Z be the meets of AC, BD, FG with the line at infinity. Let AC meet BD at P (Fig. 1). Project FG to infinity and (at the same time) the angles GAC, FAC into angles of magnitude $\frac{1}{4}\pi$. The result is shown in Fig. 2, in which small letters correspond to the capital letters of Fig. 1.

(i) $$a\{f\,n\,g\,z\}=A\{F\,N\,G\,Z\}=\{F\,N\,G\,Z\}=-1.$$

But af is perpendicular to ag, since $abcd$ is a square, hence $\alpha_1=\alpha_2$. And also $\alpha_2=\alpha_3$ (alternate angles).

(ii) $$\{a\,l\,c\,x\}=\{b\,m\,d\,y\}=-1\,;$$

and p is the midpoint of ac and bd.

Thus $$pm\,.\,py=pd^2=pc^2=pl\,.\,px\,;$$

so that l, x, y, m are concyclic.

Hence $$\angle mlp=\angle myx=\tfrac{1}{4}\pi-\alpha_3=\tfrac{1}{4}\pi-\alpha_1=\angle pan.$$

Thus lm is parallel to an: that is, l, m, n are collinear; and hence L, M, N are collinear.

L. W. H. HULL.

2527. *A permutation problem.*

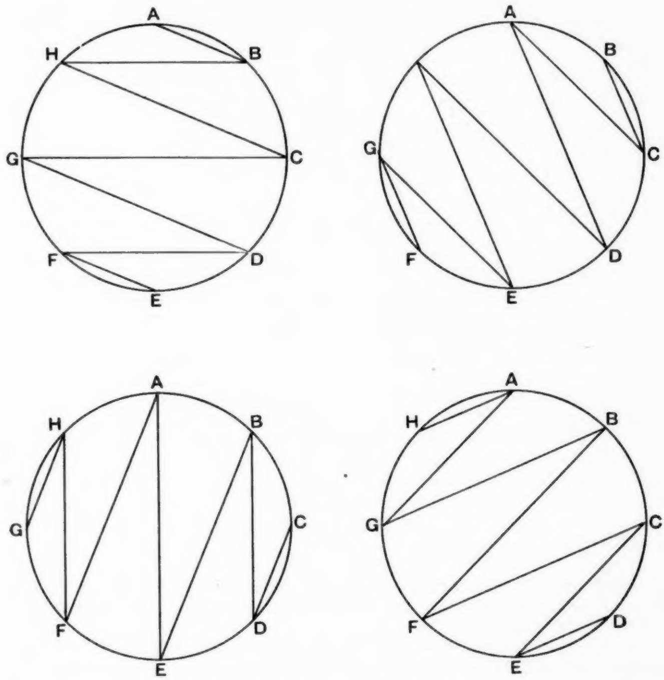
A lesson on elementary permutations produced the following problem from a pupil: "If my C.C.F. squad comprises n boys, for how many weeks can they parade in such a way that no boy has to stand next to any other boy more than once, the whole squad parading each time in a single rank?"

If we call each instance of two cadets being adjacent a "pairing", the total number of different pairings amongst n boys is $n(n-1)/2$. Since $n-1$ pairings occur at each parade, they certainly cannot parade for more than $\frac{1}{2}n$ weeks.

If n is even, the upper bound of $\frac{1}{2}n$ is always attainable. Mr. B. D. Price has sent me the following demonstration of this, in which he constructs a set of lines satisfying the conditions. The solution is shown in the accompanying

diagram for $n=8$, but it is clearly equally applicable to any even number of boys.

The order in which the points are encountered along the broken line gives the order of boys in the rank. Thus this particular solution gives the order of boys on parade for 4 weeks as follows: *ABHCGDFE, BCADHEGF, CDBEAFHG, DECFBGAH*. Each cadet will parade at the end of the line just once.



If n is odd, there cannot be more than $\frac{1}{2}(n-1)$ parades, since this is the greatest integer not greater than $\frac{1}{2}n$. The arrangements of the boys in this case may be constructed by setting down the arrangements for $n-1$ cadets and then placing the last cadet at the end of the line in each rank; he will then have a different neighbour each time.

We deduce that the solution of the problem in every case is $[\frac{1}{2}n]$.

D. A. QUADLING.

2528. *A note on models to illustrate a property of the volumes of solids.*

The areas of similar plane figures are in proportion to the squares of their corresponding linear measurements, and a variety of figures can be found such that four, nine, ... equal specimens can be fitted together to form a larger one of twice, three times, ... the linear dimensions. Some interesting and amusing

examples have been given in the *Gazette* (Vol. XXIV, No. 260, July, 1940, pp. 209–10.) It is less easy to find simple figures which illustrate the corresponding property of similar solids. Parallelepipeds and prisms do so, but only in very close relation to the two-dimensional property.

There are, however, several tetrahedra of special shapes which provide further examples.

(i) The tetrahedron $ABCD$, which has $AB=CD=a$, and all its other edges $a\sqrt{3}/2$. The faces which meet in the edges AB and CD are at right angles; all the other dihedral angles are $\pi/3$. The solid angle at each vertex is $\pi/6$. This tetrahedron can be made from the net shown in Fig. 1. If AB and CD

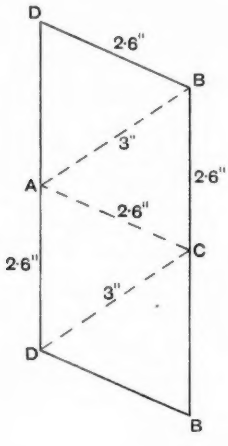


Fig. 1

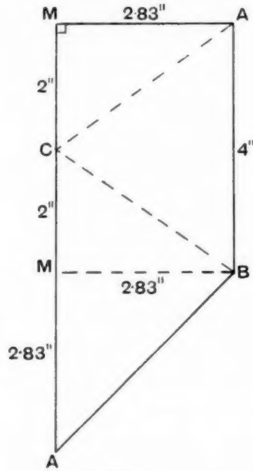


Fig. 2.

are made 3″ long, the other edges are $3\sqrt{3}/2''$ long, for which 2·6″ is a convenient approximation. 8, 27, ... n^3 of these tetrahedra may be assembled into larger similar figures.

(ii) If the tetrahedron $ABCD$ is bisected by a plane through AB and the mid-point M of CD, one of the resulting tetrahedra, $MABC$, has $AB=a$, $MA=MB=a/\sqrt{2}$, $MC=a/2$, and $AC=BC=a\sqrt{3}/2$. All the face angles at M are right angles. Numbers of specimens of this tetrahedron can also be built up into larger similar ones. Moreover, six of them can be built into a cuboid on a square base of side $a/\sqrt{2}$ and of height $a/2$. This provides an illustration of the formula for the volume of a pyramid.* The net for $MABC$ is shown in Fig. 2, with $AB=4''$.

(iii) If $MABC$ in turn is bisected along its plane of symmetry, the resulting tetrahedra have equal measurements but are mirror-images of one another. Four pairs can be assembled into a similar figure. Two identical ones and one mirror-image can be assembled into a triangular prism which is half a cube; this again illustrates the pyramid formula. Three identical ones can be assembled into an oblique prism on the same base and of the same perpendicular

* See Cundy and Rollett, *Mathematical Models*, pp. 147–8.

height as each component. Figure 3 shows the net for one half of $MABC$; N is the mid-point of AB.

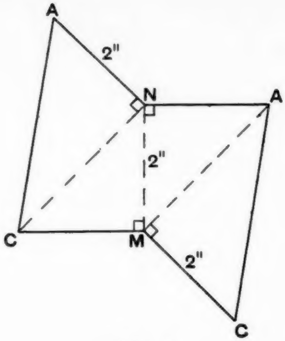


FIG. 3.

Two slightly more complicated figures retaining the same basic property may be derived from the tetrahedron $ABCD$. If two such tetrahedra are placed together with faces coinciding, a pyramid on a rhombic base results; this is one of the figures. The other is obtained by bisecting $ABCD$ by a plane through the mid-points of AB and CD and those of another pair of opposite edges.

Figures of about the sizes quoted above are most easily made up from the thin white cardboard known to stationers as No. 4 board; for larger models something stiffer such as No. 10 board is desirable. Each figure may be made up by leaving flaps of cardboard attached to the edges at suitable places, but I have found it easier to cut out the net as shown, score the dotted lines, and stick the edges together with strips of a sticky paper such as Gumstrip. I have never tried Sellotape, but useful as it is for many purposes I suspect that it might not be ideal here. Good joints can be made by creasing each strip of sticky paper before use, and applying it so that the crease falls along the edge. The absence of flaps makes it possible to draw out some of the nets as a repeated pattern with economy of both material and draughtsman's time. " Pricking through " with a pin is another useful way of getting a number of copies.

R. SIBSON.

2529. *A nomogram for the solution of quadratic equations.*

The roots of the quadratic equation

$$ax^2+bx+c=0$$

are given by the intersection of the parabola

$$y=x^2$$

with the straight line

$$ay+bx+c=0,$$

whose intercepts on the axes of x and y are $-c/b$, $-c/a$. This may be used as the basis of a nomogram as shown in Fig. 1, where the x-axis is labelled c/b, the

y-axis labelled c/a (both with reversed signs) and the values of x are marked on the curve.

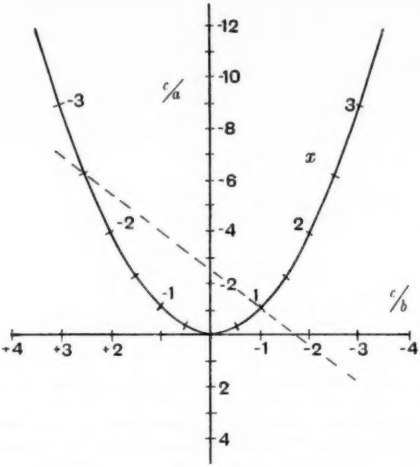


FIG. 1.

Such a nomogram has very limited application, but the useful range can be increased by projecting the coordinate axes OX, OY and the line at infinity into the sides CB, CA, AB of an equilateral triangle, and the points (1,0), (0, 1) into the midpoints of the sides BC, AC. The projection of the parabola is then a circle touching AB, BC at A and C. Using areal coordinates the equations of the curve and line become

$$\beta^2 - \alpha\gamma = 0$$
$$a\alpha + b\beta + c\gamma = 0.$$

The formerly linear scales of x and y are projected into proportional scales of γ/β, γ/α. For convenience the latter scale is reversed and graduated in terms of α/γ; AB is also graduated in terms of β/γ. A method of graduating such a scale has been described in the *Mathematical Gazette* (XXXVII, p. 39). In the nomogram, shown in Fig. 2, these scales are labelled respectively c/b, a/c, b/a. The graduations on the curve, which is the locus $\gamma/\beta = \beta/\alpha$, are obtained by projecting either the c/b scale or the b/a scale from the opposite vertex of the triangle of reference.

To keep the size of the nomogram within reasonable bounds the sides are graduated for positive values only, and only that part of the curve within the triangle is drawn. For positive values of c the nomogram is used as indicated in Fig. 3(i). The straight-edge, laid between the appropriate points on the b/a, c/b scales (the sign of b being ignored), lies along the line

$$a\alpha - |\,b\,|\,\beta + c\gamma = 0.$$

The intersections of this line with the graduated arc give the numerical values of the roots. For negative values of c the two roots are found separately as indicated in Fig. 3 (ii). Here the equations of the two lines along which the straight-edge is laid are

$$a\alpha + |\, b \,|\, \beta + c\gamma = 0, \quad a\alpha - |\, b \,|\, \beta + c\gamma = 0,$$

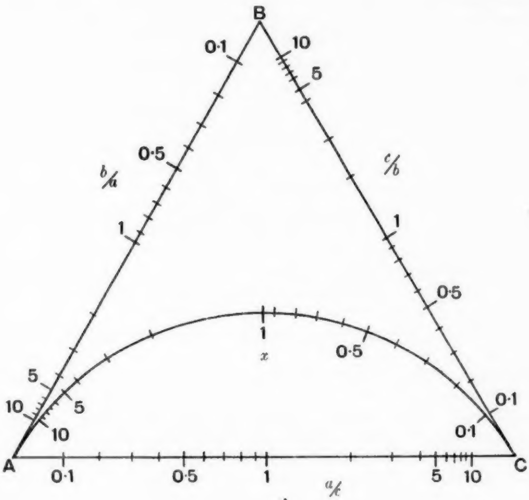


FIG. 2.

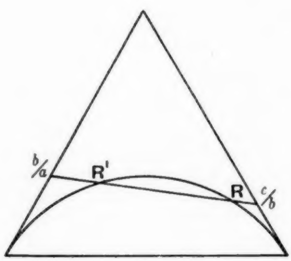


FIG. 3(i).
c positive. The roots (R, R') have the opposite sign to b.

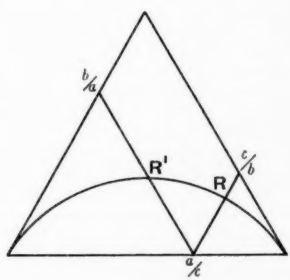


FIG. 3(ii).
c negative. The numerically smaller root has the sign of b; the numerically larger root has the opposite sign.

and again the intersections of these lines with the graduated arc give the numerical values of the roots.

C. V. GREGG.

REVIEWS

Nomological Statements and Admissible Operation. By Hans Reichenbach. Pp. 140. 27s. 1954. (North-Holland Publishing Company, Amsterdam)

In formalised logics the relation of inference between two statements is generally given a wider scope than it has in ordinary discourse. Thus in ordinary language the observation " If snow were black then sugar would be sour ", though true in most formal logics, would not be accepted as a reasonable inference ; we should feel that the lack of inward connection between the antecedent and consequent disqualify this sentence from any claim to be called an inference. Many attempts, mostly unsuccessful, have been made to formalise the intuitive notion of a reasonable inference, and the present book presents a revised version of the system set up by Reichenbach in Chapter VIII of his book *Elements of Symbolic Logic*, a revision necessitated by the adverse reception which part of the original formulation was accorded.

The attempt to explain " reasonable implication " cannot of course be confined to the single operation of implication. Disjunctions can also have " unreasonable " occurrences, which are of course only transforms of unreasonable implications, and the same is true of all propositional operations.

The implications which Reichenbach studies are not restricted to logical implications but include also physical entailment. Implications of this latter kind, express the so-called laws of nature, such as " if metal is heated it expands ", and that is why Reichenbach uses the term *nomological* implication (for both kinds).

One of the major tasks which the book undertakes is the clarification of the notion of a universal law. Hume believed that the necessity of a law of nature springs from its generality alone ; that is to say that *causal connection* differs from mere coincidence only in that it expresses a permanence of coincidence. Reichenbach considers, however, that generality alone, though necessary, is not sufficient to rule out all unreasonable inferences, and he calls a synthetic statement universal only if it cannot be written in a reduced form which contains an individual term ; the reduction process in question is, roughly speaking, that of dropping out tautologies. As an example of a general statement which is not universal in this sense Reichenbach remarks that the sentence " Every one who was the first to see the human retina contributed to the establishment of the principle of the conservation of energy " conceals a description of a particular individual (von Helmholtz).

Reichenbach introduces a hierarchy of statements with *original nomological* statements at the head, followed by *nomological* statements which comprise all statements deductively derivable from original statements. A subclass of nomological statements, called admissible statements, supplies the reasonable propositional operations, while the nomological statements themselves embrace the laws of nature and the laws of logic. Corresponding to this hierarchy of statements there is a hierarchy of truth with analytic truth as the highest, synthetic nomological truth coming next and factual truth last. The theory developed requires an elaborate technical vocabulary and an extensive examination of detail.

R. L. Goodstein.

Symbolic Logic. By Irving M. Copi. Pp. xiii, 355. $5. 1954. (Macmillan New York)

This is a first-rate introduction to the new techniques in symbolic logic, clearly written, with ample but not excessive comment, a wealth of illustrative examples, and a pleasant pedagogic style which in no way obscures the finer points of the subject.

The book falls into two parts; the first discusses logic as an instrument for testing the validity of arguments and makes full use of the natural inference techniques introduced into modern logic by Gerhard Gentzen. To illustrate this technique, and Copi's notation, we quote the proof (p. 139) of the famous syllogism: " A horse is an animal, therefore the head of a horse is the head of an animal ", which, as De Morgan remarked, all Aristotle's logic is unable to prove.

Writing Ex, Ax for " x is a horse " and " x is an animal " and Hxy for " x is the head of y ", the result to be established is that

$$(x)[(\exists y)(Ey\,.\,Hxy) \supset (\exists y)(Ay\,.\,Hxy)]$$

(that is, for every x, if there is a horse which x is the head of, then there is an animal which x is the head of) follows from

$$(x)\ (Ex \supset Ax)$$

(that is, every horse is an animal). The proof runs as follows.

$$\begin{array}{rl}
1. & (x)\ (Ex \supset Ax) \,/\, \therefore (x)[(\exists y)(Ey\,.\,Hxy)(\supset \exists y)(Ay\,.\,Hxy)] \\
2. & (y){\sim}Ay\,.\,Hxy) \\
3. & {\sim}(Ay\,.\,Hxy) \\
4. & {\sim}Ay{\sim} \vee Hxy \\
5. & Ay \supset {\sim}Hxy \\
6. & Ey \supset Ay \\
7. & Ey \supset {\sim}Hxy \\
8. & {\sim}Ey \vee {\sim}Hxy \\
9. & {\sim}(Ey\,.\,Hxy) \\
10. & (y){\sim}(Ey\,.\,Hxy) \\
11. & (y){\sim}(Ay\,.\,Hxy) \supset (y){\sim}(Ey\,.\,Hxy) \\
12. & {\sim}(y){\sim}(Ey\,.\,Hxy) \supset {\sim}(y){\sim}(Ay\,.\,Hxy) \\
13. & (\exists y)(Ey\,.\,Hxy) \supset (\exists y)(Ay\,.\,Hxy) \\
14. & (x)[(\exists y)(Ey\,.\,Hxy) \supset (Ey)(Ay\,.\,Hxy)].
\end{array}$$

In the first line, before the stroke, we have the initial assumption. In line 2 we introduce an additional assumption, (that for every y it is not true that y is an animal and x is its head), and every step from 2 to 10 uses this assumption which is finally " discharged " in line 11; the dependence of lines 2 to 10 upon assumption 2 is indicated by the ruled line on the left. The remaining steps above the line are familiar transpositions. Lines 2 to 10 constitute a derivation of 10 from 2, which by " natural inference " (or the deduction theorem) proves 11. Lines 12, 13, 14 follow by further familiar transpositions, and 14 is the desired conclusion, which is entered in line 1 after the initial assumption. Since 14 follows from 1 the final result could of course be expressed by the implication

$$15.\ (x)(Ex \supset Ay) \supset (x)[(\exists y)(Ey\,.\,Hxy) \supset (\exists y)(Ay\,.\,Hxy)]$$

which discharges the initial **implication.** Line 15 expresses a *provable formula* in the classical sense, which is not the case with any of the previous sentences. Unfortunately Copi's formulation of natural inference does not provide sufficient safeguards against false inferences, and for other than monadic predicates needs revision along the lines, for instance, of Quine's paper on natural deduction (*Journal of Symbolic Logic*, No. 15, 1950).

The second part of the book contains a study of axiomatic systems. A number of axiomatisations of the propositional calculus are considered, the first of which is attributed to Barkley Rosser, and which has not hitherto been published in this form. Other systems considered are the Hilbert-Ackermann

system, that of *Principia Mathematica*, and Nicod's system. There is a very full account of functional and deductive completeness.

The first order function calculus is set up in a formalisation which is also due to Rosser. The notions of valid and satisfiable formulae are fully explained and the calculus is proved to be deductively complete. The method of proving completeness is due to Henkin. Though of great intrinsic interest, Henkin's proof does not appear to be as well suited to an introductory text as the original proof by Gödel, nor does it yield as readily the important Löwenheim theorem that if a formula is satisfiable at all then it is satisfiable in a denumerably infinite domain.

There are two further points which perhaps merit special comment.

Copi draws a distinction between weak and strong induction (as metamathematical principles). By weak induction he means an induction which uses only $F(0)$ and $F(n) \supset F(n+1)$ to prove $F(n)$. In strong induction we use

$$F(0) \;\&\; F(1) \;\&\; F(2) \;\&\; \ldots \;\&\; F(n) \supset F(n+1).$$

The terms are taken from Rosser's *Logic for mathematicians*. Rosser, however, shows that the two principles are equivalent, whereas Copi leaves his reader under the false impression that we have two distinct principles. The derivation of the principle of strong induction from weak induction runs as follows. We have to prove the inference

$$\frac{\begin{array}{c}F(0)\\ (x)\{x \leqslant n \supset F(x)\} \supset F(n+1)\end{array}}{F(n)}$$

using the scheme

$$\left.\frac{\begin{array}{c}F(0)\\ F(n) \supset F(n+1)\end{array}}{F(n)}\right\} \qquad \text{(I)}$$

Write $G(n)$ for $(x)\{x \leqslant n \supset F(x)\}$, so that our assumptions take forms $G(0)$ and $G(n) \supset F(n+1)$. Since

$$G(n) \;\&\; F(n+1) \supset G(n+1),$$

we have

$$G(n) \supset G(n+1),$$

whence $G(n)$ holds, by scheme I. From $G(n)$ we derive

$$x \leqslant n \supset F(x),$$

whence substituting n for x, and using $n \leqslant n$, $F(n)$ follows.

Chapter III on the sentence calculus could be strengthened by the introduction of arithmetical methods. At the end of this chapter Copi says that the *reductio ad absurdum* method of assigning truth values in testing arguments, is, in the vast majority of cases, superior to any other method known, but this is only true if it is used in conjunction with arithmetical devices. Consider, for example, the tautology

$$((p \supset q) \supset p) \supset p\ ;$$

if it is false, Copi argues, then $(p \supset q) \supset p$ is true and p is false, whence $p \supset q$ is true and p is false, and finally p is true which is a contradiction.

If we let p and q denote variables which take only the values zero and unity, $p = 0$ expressing the truth of p and $p = 1$ the falsehood of p, then the negation of p, the disjunction of p and q, their conjunction, and the implication $p \supset q$ are represented respectively by the functions $1-p$, pq, $p+q-pq$, and $(1-p)q$.

Thus

$$((p \supset q) \supset p) \supset p$$

is represented by

$$(1 - (1 - (1 - p)q)p)p.$$

If this function takes the value unity, then $p = 1$, $1 - (1 - p)q = 0$ and so $(1 - p)q = 1$, whence $p = 0$; this contradiction shows that the value of the function is zero for all p, q.

Similarly, to prove the validity of the argument

$$(A \vee B) \supset (C \,\&\, D),\ (D \vee E) \supset F, \text{ therefore } (A \supset F),$$

we have to show that from the equations

$$(1 - ab)(c + d - cd) = 0, \quad (1 - de)f = 0$$

follows $(1 - a)f = 0$; it suffices to observe that if $(1 - a)f = 1$, then $a = 0, f = 1$, whence $de = 1$ and so $d = e = 1$ and $c + 1 = c$, which is impossible.

R. L. GOODSTEIN

Mathematics in Action. By O. G. SUTTON. Pp. viii, 226. 16s. 1954. (Bell)

The theme of this book is the part played by mathematics in applied science. In a chapter on " the tools of the trade ", Dr. Sutton gives a brief introduction to complex numbers and calculus, including total and partial differential equations. He stops short of tensors and decides against the use of vector notation, as demanding experience in the reader for its comprehension. In fact the layman, for whom the book is primarily written, is likely to find the mathematicial symbolism difficult, but he can appreciate the general picture of the solution of physical problems, often idealised or simplified, by means of mathematical equations.

The particular topics discussed at some length are ballistics, waves (including a descriptive account of wave mechanics), aerodynamics, statistics and meteorology, all of them fields in which the author has worked. An example is given in ballistics of the numerical calculation of the trajectory of a shell at intervals of one second from a differential equation which cannot be integrated. Modern computing machines will relieve us of the numerical drudgery, but there is much scope for mathematical knowledge and ability in the coding of problems. Moreover, the " mathematical technician " must understand and speak the language of the electronic engineer. It appears, however, that even electronic machines might fail to " keep up with the weather ", had we the data and equations necessary for accurate forecasting.

A minor criticism is that the use of dy for δy on p. 45 is not consistent with the definition of the differential on p. 42. It should be explained, too, that in a test of significance using the binomial distribution, the relevant probability is the sum of the probabilities of the event tested and of all equally or less probable events. Even the layman might object, on mathematical grounds, to the statement that " seven heads and three tails . . . is expected . . . in at least three out of ten trials of ten tosses each ".

Besides the layman, the author hopes to interest students in the upper forms of schools or in their first year at a university. From the point of view of the teacher of mathematics, the book may help to promote incentive, either by dispelling the idea, if it still persists, that specialisation in mathematics can lead to no other career than teaching, or by inspiring the type of pupil whose interest in mathematics is dormant until he realises the power and importance of the subject in practical applications. For this reason, as well as its general interest, the book is worth a place in the library of a grammar school or technical college.

C. G. P.

Differential Operators and Differential Equations of Infinite Order with Constant Coefficients. P. C. Sikkema. Pp. 223. Fr. 11.50; cloth, fr. 13.50. 1953. (Noordhoff, Groningen)

This is, substantially, the author's doctoral thesis. Contents : Preparatory Chapter ; Introduction ; I. Necessary and Sufficient Conditions ; II. Properties of the Function $h(x)=F(D)\to y(x)$; III. Further Investigation of the Function $h(x)=F(D)\to y(x)$; IV. On the Differential Equation $F(D)\to y(x)=h(x)$; V. Chapters III and IV Continued.

Here $F(x)$, $h(x)$, $y(x)$ are integral functions of the complex variable x, and the operator " $F(D)\to$ " is interpreted formally in the obvious way (it being one of the author's preoccupations to determine when the series $F(D)\to h(x)$ converges). The author imposes conditions on two of $F(x)$, $h(x)$, $y(x)$ and considers what can then be said (e.g. in terms of " type " and " order ") about the third ; uniqueness theorems are proved for $y(x)$. His results are new and interesting, but his style is excessively diffuse and repetitive. The printing is good.

M. P. D.

Differential Line Geometry. By V. Hlavatý. Translation (based on the German text) by H. Levy. Pp. x, 495. D. fl. 22.50 ; cloth fl. 25. 1953. (Noordhoff, Groningen)

The original Czech edition of this book was published by the Czech Academy of Sciences in 1941. A German translation by M. Pini was published by P. Noordhoff in 1945. The present English translation by Professor H. Levy is based upon the German text. Apart from numerous minor corrections and modifications throughout the whole book, the main difference between this and the German edition occurs in sections of chapter 9 dealing with irregular lines of congruences, and in certain sections of chapter 10 dealing with anallagmatic geometry.

It is well-known that by using its Plücker coordinates a straight line can be represented by a point on a four-dimensional quadric K in a five-dimensional projective space. This book is essentially a treatise about the differential geometry of the quadric K and its submanifolds, and its application to the study of properties of the corresponding line-manifolds. This method of investigation has been used previously in papers by W. Haak, K. Takeda and J. Kanitani ; but the method is developed here in great detail and in a manner which allows the simultaneous study of projective, affine and metric properties. The methods of tensor calculus are used throughout the book, and a carefully written appendix of 28 pages summarises those parts of the subject most frequently used.

The book is written in five parts, each of which can be read independently of the others. The first part (Chapter I) deals with the representation of lines by points of the quadric K. The second part (Chapter II) is concerned with ruled surfaces—these correspond to curves on the quadric K. The third part (Chapters III, IV and V) is devoted to the study of congruences. The fourth part (Chapters VI, VII, VIII and IX) deals with complexes. The fifth part is concerned with the study of line-space, including the derivation of the fundamental equations of Weingarten, Gauss and Mainardi-Codazzi.

The book is clearly written and contains a number of exercises which illustrate the text. There is a very useful index at the end of the volume. There is no doubt that the book will remain one of the standard works on this rather specialised subject for many years to come.

T. J. Willmore.

Axiomatic Projective Geometry. By R. L. Goodstein and E. J. F. Primrose. Pp. xi, 140. 15s. 1953. (University College, Leicester)

This book, intended for university students taking a first course in projective geometry, covers the axiomatic plane geometry of the new syllabus for the London University Special Honours Degree in Mathematics. There are ten chapters entitled " Homography ", " The harmonic construction ", " The conic ", " Amicable involutions ", " The fundamental involution ", " Complex geometry ", " The circular points ", " The common points of two conics ", " The reduction of a chain of perspectivities ", and " Geometry as a board game ". Each of the first eight chapters ends with a set of examples, and detailed solutions to these are given on pages 124–138.

The treatment is purely synthetic, and coordinates are not used at all. Axioms are introduced as required in the development of the subject ; these are conveniently collected for reference on page xi, together with the chapter number in which they are introduced. The authors have succeeded in their task of showing that the subject can be developed rigorously and elegantly, while keeping the arguments at an elementary level.

Chapter 9 contains a proof of the result that if ranges on two lines are homographically related, the sequence of perspectivities connecting them may be reduced to two if the ranges are on different lines, and reduced to at most three if they are on the same line. In the latter case it is proved that the sequence can still be reduced to two provided that there is at least one self-corresponding point in the homography. This condition is often glossed over in text-books on elementary projective geometry.

It is unfortunate that, although the text is well printed, some of the eighty figures in the book are not carefully drawn. Not only does the thickness of line vary from figure to figure, but there is often considerable variation in the thickness of lines in the same figure. Moreover, sometimes the argument is obscured by the presence of a figure. For example, the diagram on page 16 is intended to illustrate a system of seven points and seven lines such that a unique line passes through each pair of points, and such that each pair of lines meets at a unique point. Most students reading the subject for the first time will surely find that the diagram makes matters more difficult.

The last chapter illustrates the formal nature of the subject, and will no doubt be found difficult by many students for whom the book is intended. Nevertheless such students will learn much from other parts of the book, especially from the collection of over 120 examples and the fifteen pages giving their solutions.

T. J. Willmore.

Théorie des Fonctions Aléatoires. Applications à Divers Phénomènes de Fluctuation. By A. Blanc-Lapierre and R. Fortet. Pp. xvi, 693. Paper, 6,000 f. ; cloth boards, 6,500 f. 1953. (Masson, Paris)

This treatise on the theory of " random functions " (i.e. of " random variables " in English, although " random functions " is the more appropriate) is a happy collaboration between a physicist and a mathematician. It gives a comprehensive account of the modern theory of stochastic processes and of the applications of the theory to the analysis of the many types of time-series which may arise in practical problems. This book is important to British readers because there is no work in English of equal range and generality which combines both theory and application. (Doob's *Stochastic Processes*, reviewed in *Math. Gaz.* XXXVIII, No. 325 is comprehensive but is purely mathematical).

The treatise begins with two introductory chapters. The first describes the

kinds of physical process to which the theory can be applied; it is intended mainly for physicists and engineers. The second is a summary of the relevant concepts and theorems of probability theory, including results established in recent years in France (mainly by Lévy and Fréchet) and in America (mainly by Feller).

The development of the theory occpies the next ten chapters. Though the development is mathematically rigorous, the authors are interested in physical applications of the theory and so the mathematical results are usually interpreted and discussed in physical terms. To help the reader, those sections which are likely to be of interest only to mathematicians, and which can be safely omitted by scientists on a first reading, are marked by one asterisk; those mathematical sections which have only limited application are marked by two asterisks.

The main theory begins in Chapter III with a very general mathematical account of random functions which enables the authors to establish their terminology and notation. In the next chapter stochastic processes are described and broadly classified. First consideration is given to processes with independent random increments. There is a detailed analysis of the Wiener-Lévy process, in which the random function $x(t)$, continuous in probability, has, in the reduced case, random increments $x(t+\Delta t)-x(t)$ which are normally distributed with expectation zero and variance Δt. There follows an analysis of stationary Poisson processes, and of processes closely related to them, which, because of their wide physical occurrence, is further developed and generalized in Chapter V with many physical illustrations.

There follow two chapters on Markov processes which are described in terms of a physical system which is evolving towards a state of equilibrium but which is affected by random small perturbations displacing the system from the state of equilibrium it is tending to attain. The basic physical picture used in illustration is that of the level of a reservoir which is both receiving water in the form of rain-drops and is providing water at a rate which is a function of the height of the level above the outflow orifice. In such processes the random variable $x(t+h)-x(t)$, where h is positive, is independent of the history of the system (except insofar as the history is summarized in the value of $x(t)$) but is dependent on the final state of equilibrium towards which the system is tending. In these two chapters the mathematical theory of Markov processes is not only developed and generalized but is very fully illustrated by examples drawn from a wide range of phenomena. This discussion includes, for example, close analysis of the purely discrete process (Feller) and of the purely continuous process (Kolmogorov).

The next five chapters constitute a comparatively formal and conventional development of the theory of time-series though the theory continues to be enlightened by frequent physical interpretations. In these chapters the several forms of stationary time-series are closely analysed and the harmonic analysis ergodic properties of the series are discussed. The authors here provide a coherent logical account of many contributions made to the theory in recent years by European and American mathematicians and which hitherto have had to be sought in papers scattered over many journals and in several languages.

Chapters XIII and XIV are the only two chapters devoted entirely to specific applications. The first of these, much the shorter, is on the background noise that is of special concern to telecommunications engineers and to physicists who have to observe "weak signals" (not necessarily audio-signals) from physical systems. Looking back on the formidable array of the previous twelve chapters, this chapter is perhaps a little disappointing; for example, one might have expected some application of information theory

to the problems of background noise but the chapter ends abruptly with the mention of Hartley's fundamental equation for the entropy of a noisy channel.

The second of these two chapters is written by a third author, Professor Kampé de Fériet, on the subject of the Statistical Theory of Turbulence. This theory, founded on two early papers of Sir Geoffrey Taylor, has been rapidly extended in recent years by applying to it the theory of random functions. It is surprising to find that Professor Kampé de Fériet disengages himself from the theory of random functions as it is developed earlier in the book by his co-authors ; he relies more on Wiener's representation of a random function as an *ensemble* of functions with random parameters than on the axiomatic approach of Kolmogorov and Doob which is based on measure theory. Though it has been shown that these two approaches can be reconciled, Wiener's is the more readily applicable to physical problems.

It is encouraging to find that the authors of this treatise find it necessary even for French readers, to conclude with a chapter which is in effect an appendix summarizing measure theory insofar as it is required for understanding their development of the theory of random functions.

The publishers express the hope that this work may be read by undergraduates in this country, but the pre-requisites to its reading are a knowledge of measure theory, of modern probability theory and of physics, a combination which can only rarely be found here in courses for first degrees. Yet in the theory of random functions there is a field of study which is receiving the concentrated attention of mathematicians elsewhere, a field to which British workers have contributed little, yet which is increasing rapidly in both theoretical and practical importance. Although the French text is not difficult to read and the theory is very fully expounded, there are differences of terminology between French and American writers which might confuse some English readers. A translation into English, preferably a more concise version of selected parts of this treatise, would help to arouse more interest in this subject in this country.

It is pleasing to see that the publishers of this interesting French treatise have given it a worthy format ; the printing and binding are excellent.

B. C. BROOKES.

Elements of Statistics. By H. C. FRYER. Pp. viii, 262. 38s. 1954. (Wiley, New York ; Chapman and Hall, London)

This American text-book provides a first course in elementary statistics and probability theory for students who need statistics as an ancillary. It is based on a course of lectures and practical work that has been evolved in recent years by the author at Kansas State College.

The scope of the book is restricted to a descriptive account of frequency distributions, the Binomial and Normal probability distributions, sampling from Binomial and Normal populations, linear regression and correlation. The necessary tables are included.

The treatment is sound and thorough ; it is evidently based, as the author claims, on wide teaching experience, and it should lead the readers for whom it is intended to an intuitive understanding of the subject. The text is well-written in a style which is likely to be appreciated by those who prefer words to mathematics. Though the scope of the book is elementary the author introduces some more advanced ideas with good effect ; thus the early introduction of the concept of *mathematical expectation* and the discussion (with no proof) of the implications of the *central limit theorem* are helpful. There are many general and numerical exercises, to half of which answers are given.

Though some of the diagrams (for example, those on pages 91, 93 and 157) are rather crudely drawn and could be misleading, these are the only defects

in a useful book which otherwise reaches the high standard of presentation set by the earlier volumes in the Wiley Publications on Statistics. Unfortunately the price of this book is very high for what it offers to potential British readers.

B. C. BROOKES.

Exercises in Theoretical Statistics. By M. G. KENDALL. Pp. vii, 179. 20s. 1954. (Griffin)

Most teachers of statistics laboriously collect together from diverse sources a heterogeneous collection of exercises suitable for their students, and have long vaguely felt the need for some richer source of material than text-books provide. Professor Kendall has evidently set about the task of collecting exercises systematically and comprehensively, and, having done so, he has generously presented other teachers with the results of his work.

The book contains 400 classified and graded exercises ; some are drawn from university examination papers, some from research papers, and many are doubtless invented by the author. All are given answers, most are provided with hints for solution, and, when it is applicable, a reference is given to papers which offer further help and understanding. The exercises consist of 100 on Distribution Theory, 100 on Sampling Theory, 75 on Statistical Relationship, 75 on Estimation and Inference, and 50 on Time-Series. They are all mathematical rather than numerical, and all relate to basic theory. Within each section they are graded in difficulty and cover the whole range of university diploma and degree courses.

It soon becomes evident to the user that the questions have been selected with care and graded with understanding of the student's difficulties. The hints are adequate and will be especially helpful to private students. The book is well printed and produced ; no misprint has been detected in the exercises so far used by the reviewer. This collection of exercises is likely to establish itself quickly as an essential part of the equipment of every university student and teacher of statistical theory.

B. C. BROOKES.

Advanced Level Pure Mathematics. By S. L. GREEN. Parts I and II. Pp. iv, 264, iv. 6s. each. 1953 (University Tutorial Press)

Advanced Level Pure Mathematics by S. L. Green provides in Part I the co-ordinate geometry of the straight line, pairs of lines, the circle and the conics, devoting a self-contained chapter to each topic. Some of the more important results are obtained for the conics by Euclidean geometry, and for the ellipse in particular by orthogonal projection. The bookwork is of the type likely to be encountered in A-level papers, and is presented in a manner usually very easy to follow. Of the examples, many are numerical, but there is also a large selection to exercise the manipulative and logical faculties. The printing and diagrams are good and clear, but the hyperbolas are often curves which can obviously and uncomfortably be cut by numerous straight lines in four points ; indeed, one—Fig. 28—contains a line through one focus *touching* the other branch of the curve !

Part II contains a useful and concise selection of theorems in the geometry of the triangle and the solid geometry of planes and spheres, and ends with a development of trigonometry from ratio-definitions applicable to angles of any magnitude. This development is entirely geometrical, its aim being to establish relations between angles and distances. The detailed instructions for the use of 4-figure tables seem superfluous for a student at this stage and the same can be said of the solution of a triangle given three sides, by half-angle formula, where the third angle is lazily and incorrectly determined by the angle-sum of a triangle. The examples are well-chosen and numerous, many

of them being from A-level and Scholarship papers, and the pupil for whom the books are intended should be able to attain a grasp of their methods with the minimum of outside assistance. W. J. H.

The Geometry of Mental Measurement. By Sir Godfrey H. Thomson. Pp. 60. 6s. 6d. 1954. (University of London Press)

Psychometrists have to handle a number of measurements, one for each of a series of tests, for each of their subjects. Such measurements are usually found to be correlated, and the question arises whether the measurements can be represented by an equation in fewer unknowns than there are tests. A number of solutions to this problem have been offered, and British workers in the field of " factor analysis " include, besides the names of C. Spearman, Cyril Burt, and, more recently, P. Vernon, that of Professor Sir Godfrey Thomson. Thomson was for long fighting, from the University of Edinburgh and from Moray House in that city, the claims of his solution. He has now retired from his chair and the present book has its origin in three special University Lectures given in the Senate House of the University of London, in January, 1953. Its primary purpose is stated by the author to " describe a geometrical model from which can be deduced most of the formulae used in the factorial analysis of human ability ".

The model given requires " only a very meagre mathematical equipment from the reader " and it can be regarded as a method of avoiding spherical trigonometry in n dimensions. To do this, Thomson starts with the similar and similarly situated ellipsoids that serve as contour surfaces of equal frequency for the measurements in n-dimensional space of the various subjects on the various tests, the test axes being orthogonal. He first changes the scales of the axes so that the s.d. of each test is unity : this gives the ellipsoid with its axes equally inclined to the test axes. He then distorts the ellipsoid to get a sphere in n-dimensional space, in such a way that by so doing he gets the lines of the tests to be inclined to one another at angles the cosines of which are the correlations between each pair of tests.

There are now two methods by which the required common factors are found. In each the algebraic procedure consists in working from the matrix of correlations between every pair of tests, and for each Thomson gives a geometrical analogue.

In the first method, each entry in the diagonal of the matrix, that is, the correlation for $i=j$, is put as 1. The tests are then regarded as unit vectors. They have a resultant. This is here called the centroid, and the cosines of the angles of the test with this " centroid " give the " loadings ", that is, the saturation of each of the tests with the centroid factor. The components of the tests in the $(n-1)$-dimensional space perpendicular to the centroid are now vectors in equilibrium. Thomson does not adopt any bipolar factors here, but, by what seems an unsatisfactory device, reverses the signs of as many of the components as will give the narrowest pencil of vector components. He then finds the centroid of this. And so on, in subspaces of $(n-2)$, $(n-3)$, . . . dimensions, until n orthogonal factors have replaced the n oblique test lines.

The other method inserts in the diagonal what are termed " minimum communalities ", these being guessed values, and finding factors as before. The effect of this is equivalent to having a space of $(n+c)$ dimensions divisible into a common factor space of c dimensions (where c is the number of common factors found) and a specific factor space of n dimensions. The test space, also of n dimensions, is part of the $(n+c)$ space, but it neither coincides with the specific factor space nor contains the common factor space. Thomson does not here deal with the question of how far to go seeking factors. The number of

them, c, depends on the estimate of the s.ds. involved (for as every measurement is subject to " error " no series of fewer than n common factors can give values exactly equivalent to the original matrix of measurements): the guessing of the values of the communalities is intended to make c a minimum. It may be noted that Thomson looks on each test line as having $c+1$ co-ordinates, c in the common factor space and 1 along the specific axis belonging to that test. In other words, Thomson has not adopted the idea that there is such a thing as a " group " factor, a solution preferred by some other workers in this field. The treatment does not deal with the modern attempts to use non-orthogonal common factors (a procedure, one, I think, of doubtful validity and utility, indicated by Thomson in another of his recent booklets by the analogy of trying to find, for a number of cuboids of various measurements of length, breadth, and height, the common features that they possess in respect of size and shape). It does deal with the application of the procedure to the question of prediction and of " selection ", though it is not easy to see what kind of selection is being made in section 2 when the average score is not altered, the s.d. is reduced, and the distribution remains normal.

Thomson's treatment here has clarified some of the ideas of the factor analysts, though, as indicated above, some of those that seem still to require justification are not yet put convincingly. But all who are interested in these modern attempts to find common statistical features in series of measurements that arise in cases such as those of the teacher, the students of bodily build, the students of rheology, and students of similar problems, should read this little work by one of the leading British research workers on such problems.

FRANK SANDON.

Essential Calculus. By R. W. STOTT. Pp. 80. 3s. 3d. 1953. (University of London Press)

This contains the Calculus essential for students taking Physics to scholarship level in the G.C.E. It can be covered in two terms at the rate of two teaching periods and a homework per week.

There are a few minor blemishes which can easily be remedied, e.g. confusing notation in § 3, p. 23, no apparent justification for the transition from $\Sigma(mr^2)$ on p. 64 to the result for dI/dr on p. 65, and a wrong value for angle PQN on p. 27, but the treatment is sound, clear, and suitably informal, and the examples are sufficiently numerous and frequent. The book should be particularly useful in Sixth Forms where pupils doing Physics and Biology cannot also do Mathematics to advanced level.

A. H. G. P.

A General Certificate Calculus. By L. HARWOOD CLARKE. Pp. viii, 222. 10s. 6d. 1953. (William Heinemann)

This book is intended for pupils beginning Calculus in the Sixth form and taking Pure Mathematics to advanced level in the G.C.E. Differential equations are included, but not curvature, asymptotes, or partial differentiation.

The needs of the examination are well met, and many of the examples are from G.C.E. papers. The sections on maxima and minima, the inverse trigonometrical functions, the logarithmic and exponential functions, and arcs deserve particular praise. That on areas has all the necessary material, but the argument lacks cogency. Throughout the book, indeed, the author seems happier with symbols than with words. The only technical inadequacy noticed was in the treatment of differentials. These are introduced (p. 40) by saying that $dy/dx=x+1$ may by common convention be written $dy=(x+1)dx$, and that $2x \, . \, dx$ is called the differential of x^2. This certainly does

not justify the inference on p. 45 of $dS = \pi(2h\,.\,dr + 2r\,.\,dh + 2r\,.\,dr)$ from $S = 2\pi rh + \pi r^2$.

The book is well produced, with clear print and an excellent binding, and is good value for the money. A. H. G. P.

Man and Number. By D. SMELTZER. Pp. 112. 7s. 6d. 1953. (A. & C. Black)

In this book Mr. Smeltzer has traced the history of Number from the vague number sense of early man to counting and reckoning as known at the present time.

Although much of the matter will be familiar to readers of such books as Dantzig's *Number, the Language of Science* and D. E. Smith's *Number Stories of Long Ago* the style and gradual development of the theme makes this an attractive book.

The chapter on recording is particularly good and gives general descriptions of the methods used in many countries.

There are some interesting tests for the reader on his visual sense of number, and the old names for the numbers from 1 to 20 as used in the Yorkshire Dales make delightful reading.

After describing the various calculating devices on fingers and counting boards, the author emphasises the importance of the Hindu system of numbers which combined a concise symbolism with the principle of place value. In conclusion, he considers the inconsistencies and disadvantages in our present number system.

Some readers may find such statements as $21 = 1\ 1 + 0\ 2s + 1\ 4 + 0\ 8s + 1\ 16$ a little difficult to follow, but otherwise the text is clear and attractive and the book should find a place in primary and secondary schools alike. It not only tells an interesting and human story but it clearly shows the inherent difficulty man has had in dealing with number. W. A. C.

Arithmetic Made Easy. By W. HADYN RICHARDS. Book II. Part 1. Pp. 96. Part 2. Pp. 95 and two charts. 3s. each. 1953. (Harrap)

Book 2 continues the method of stating a rule, and then setting a practice exercise on it. The directions are addressed to the pupil, who for the most part is intended to go on without other help, but is occasionally told to refer to his teacher. Brighter children of 8 or 9 should be able to follow the instructions, and these might be of considerable help to older boys and girls who had become conscious of difficulties to be " made easy ". Care has been taken to include all the basic facts, and to provide opportunity for them to be applied in a wide variety of " sums with words ", and there are ingenious exercises to make sure that signs, and technical words like " difference " and " product " are being correctly used. These books complete the usual multiplication and division tables, and introduce simple work with money, length, capacity, weight, and time. The method used in subtraction is that of " equal additions ", but it is not quite consistently applied—since in $37 - 19$ the pupil is directed to say " add ten to the 7, 9 from 17," but in $3/5 - 1/9$ he is told to say, " add a shilling to 5d., 9d. from 1/- is 3d., and 5d. makes 8d." (not " 9d. from 17d. makes 8d.").

There are a few places where an instruction is given at the foot of the right-hand page, and the figures to which it refers are on the back of it, and this is liable to create a difficulty unnecessarily. Moreover, in Part 2, in the middle of p. 14, the instruction is by no means clear, and on p. 42 the heading " Mixed Sharing " is open to criticism—" Practice in Division " would be better

H. M. C.

Applied Descriptive Geometry. By F. M. WARNER. 4th edition. Pp. viii, 247. 32s. 1954. (McGraw-Hill)

An American text book and hence all projection is in third angle. It is well printed on good quality paper with excellent binding. The price is rather high by English standards. Text is clear and simple to follow. The material is of approximately the standard required for National Certificate S.2 or B.Sc. Eng., drawing, i.e. Orthographic projection, auxiliary views, point line and plane problems, revolution method, concurrent non-coplanar forces, curved lines and surfaces. The chapter on non-concurrent forces is particularly interesting. An exceptionally large number of problems are included, most of which have a practical bias.

P. A. E. S.

Der Pythagoreische Lehrsatz. By W. LIETZMANN. 7th Edition, 1953, pp. 92, 73 figs., bibliography and index. 3.60 DM.

Wo Steckt der Fehler? By W. LIETZMANN. 3rd Edition, 1953, pp. 184, 121 figures ; revised and enlarged, with index.

Riesen und Zwerge im Zahlenreich. By W. LEITZMANN. 5th Edition, 1953, pp. 59. 9 Diagrams, with index. 2.40 DM. (B. G. Teubner, Mathematisch-Physikalische Bibliothek, Stuttgart.)

Here are three welcome resurrections, in new and stouter jackets. In all are alterations and additions, as might be expected after a lapse of some ten years.

Of writers on topics in the elementary fields (to which he does not confine his art) surely Professor Lietzmann holds the palm. Indeed, such is his fecundity and felicity that one might term it sleight of hand : for the new reader of *Der Pythagoreische Lehrsatz* he produces rabbit after rabbit from his hat. From simple dissections, not only Perigal's, to Fermat's famous hypothesis the whole gamut is run. There are two criticisms, one trifling, the other serious. On p. 36, in Fig. 41, is the "Hilfsdreieck" really necessary? Secondly, why, why, Oh Herren Lietzmann and Teubner, have you omitted President Hindenburg's sketches of Pythagoras Vor und Nach?

Wo steckt der Fehler? is another Lietzmann classic which should be on every teacher's most convenient shelf. Dr. Gattegno bids us learn from our pupils' mistakes, and here in abundance are mistakes colossal, intriguing and instructive. It is more, however, than the usual list of paradoxes, illusions and plain errors : the professor casts his net wide, and in a short section on "autorenfehler" has brought up some very queer fish, to encourage, he says, our erring pupils. First on this list comes a passage on functionality of a truly astonishing turgidity. No prizes are offered for solution.

In *Riesen and Zwerge* the Professor is back in his rich, discursive vein. He writes so charmingly about numbers because he is fond of them. All is grist to his mill : the Sandreckoner, of course, but also apt comments on German faults in numeration, such as "acht und zwanzig" instead of "zwanzig-acht" ; astronomical and atomic units ; permutations, convergent series and, naturally, Shanks his π. Rejoice, too, at this :

"The Yancos of the Amazon have for 3 the word Poettarrarorincoaroac. 'God be thanked their arithmetic stops at that,' quoth my informant."

This book, or its English counterpart, would surely be welcomed in this country. Professor Aitken, please note : et tu in Arcadia vixisti. In his preface Dr. Lietzmann writes "Es soll dadurch mithelfen, Zahlverständnis und Zahlanschauung zu finden : beides tut uns bitter not." And so, I think, say all of us.

J. E. BLAMEY.

Das Delische Problem. By W. Breidenbach. 3rd Edition, pp. 57, 34 diagrams, index. 2.40 DM. 1953. (B. G. Teubner, Math.-Phys. Bibl., Stuttgart)

In this small book Dr. Breidenbach pursues his theme with skill and sound scholarship. Some might regret the exclusion of Archytas' dramatic tour d'esprit, but it would be out of context. Following a short history of the problem, the Professor defines and discusses " exact " and " elementary " solutions. Gauss and Hippocrates begin with statements of the problem, which are blended to form the basis of a good selection of both types of solution from Apollonius of Perga (265 B.C.) to Longchamps (A.D. 1888). The thesis is then brought to a fitting climax with Gauss' celebrated proof of the impossibility of elementary solution. With this the wind is nicely tempered to us shorn lambs : the introduction, with a definition of rational and irrational numbers begins on p. 36 : the proof occupies one and a half pages, Nos. 56 and 57.

This book is an attractive and lucid account which emphasises the intense satisfaction afforded the human mind in the pursuit of the unattainable. I cannot help quoting once again : Professor Breidenbach closes with " so können wir uns doch freuen, dass es überhaupt gelungen ist, den Unmöglichkeitsbeweis zu finden."

J. E. Blamey.

Theorie und Praxis des logarithmischen Rechenstabes. By A. Rohrberg. 11th enlarged edition, pp. 64, 15 figures. 2.50 DM. 1953. (B. G. Teubner, Math.-Phys. Bibl., Stuttgart)

Professor Rohrberg's book confines itself in the main to slide rule technique, and the " theory " part of the title is barely justified. The uses of the instrument and its manipulation are explained in a comprehensive and thorough manner. Last and by no means least comes a short history of the slide rule, and it is remarkable that out of the nine pioneers named, five are British, whereas in *Das Delische Problem* out of thirteen solutions only one is by an Englishman. Are we in these islands over-inclined to the practical? The book ends with a summary of the principal settings and a list of the best known types of rule. Finally, has anybody ever summarised Gunter so neatly?

" dem englischen Theologen . . . der sich aus Liebhaberei mit der Mathematik beschäftigte."

J. E. Blamey.

Leerbook der Vlakke Meetkunde. By Dr. P. Moelenbroek. 11th Edition, revised and rewritten by Dr. P. Wijdenes. Pp. 629 with index, and lists of formulae and contents. Limp cover f. 15.00 ; boards f. 17.50. (P. Noordhoff, Groningen and Djakarta)

This is a magnificent book : well written, clearly composed and beautifully bound and printed. I do not know when the 1st edition appeared, but my guess is the middle of last century, and the whole publication contrasts with English dependence on the form, content and style of Euclid. It is intended to follow and amplify a school geometry, and corresponds to a first and second year university text book in England, though much of its content has now disappeared from all but the specialist courses here. The first part is therefore a clear presentation of the axiomatic foundations, and the deductions therefrom ; the subsequent treatment and sequence is broader than the average English text, and rather more comprehensible on that account : the landscape is not obscured by details of leaf structure.

After this recapitulation of basic ideas, the themes broaden and particularise : theorems of Menelaus and Ceva, harmonic ranges, pole and polar, radical axes, inversion and maxima and minima. There is, therefore, no difference

from the usual English treatment on content, but in style and breadth of view the book is superior. There is, for instance, a chapter on regular polygons which culminates in the geometry of circular arc and area, with comment, relevant and irrelevant, but wholly delightful, which English writers leave to such rare commentators as F. C. Boon. Following Dr. Wijdenes' very welcome custom, the concluding chapter, in this case on the logical foundations of geometry, is by a "guest artist," K. Harlaar, who has summarised van der Waerden's book. There follows a lengthy and attractive list of potted biographies of distinguished geometers through the ages, compiled by Dr. Dyksterhuis, and the book finishes with an index, a list of formulae and summary of contents.

I cannot recall any English publication quite as good as this in the field of elementary mathematics. Of recent years, of course, deductive plane geometry has been in eclipse, with schools and universities rather favouring the early acquisition of advanced techniques. This may be a mistake, as bad a mistake in its way as that of earlier generations who insisted too much on Euclidean minutiae and inflicted his austerity of logic on immature minds. The attraction of this book is that it is not merely on geometry, but about mathematics. With strict adherence to a syllabus ignored, so many attractive by-ways gain a place: the proof, for instance, of Morley's Triangle; a theorem, named after Wallace, for calculating the bisector of an angle of a triangle, with a rider to this giving a simple arithmetical proof of the biggest practical joke of elementary geometry; two simple constructions, due to Kochansky and Mascheroni (opus citatur!) for approximations to the circumference of a circle. The list could be made longer.

Even the best of books has its faults. Perhaps this one is inclined to tell the student too much. An instance may be cited on p. 279, where the 30°, 60° triangle has 3 diagrams, one with sides a, $\frac{1}{2}a$, $\frac{1}{2}a\sqrt{3}$; the second with $2b$, b, $b\sqrt{3}$ and yet a third with $\frac{2}{3}h\sqrt{3}$, $\frac{1}{3}h\sqrt{3}$ and h. Surely one would suffice: if the reader cannot supply the others, he should not be a reader of this book. Perhaps, too, in a course of geometry, a geometrical treatment of "the extensions of Pythagoras" might be preferred. J. E. Blamey.

Vlakke Meetkundige voor voortgezette studie. By P. Wijdenes and J. Best. Pp. 301 with index and lists of formulae and contents. Limp covers f. 13.00, boards f. 14.50. (P. Noordhoff, Groningen and Djakarta)

This is an abridged edition of "Moelenbroek," written for students who have less study time or aptitude for the assimilation of the larger book. The authors have omitted wisely and committed with judgment: many proofs, for instance, are modified with deft, slight touches such as only a skilled teacher can provide for less mathematically-minded students. It is interesting to compare sections of the two books, alike in form and content, to realise that there is art and virtuosity in the compilation of a text book. Both books are well provided with graded sets of exercises. It is inevitable that the magnificent chapter, on the regular polygons and the circle, of "Moelenbroek" is ruthlessly shortened in the smaller book, but even the rump that is left makes good reading—and excellent mathematical sense. J. E. Blamey.

Tables of 10^x (Antilogarithms to the Base 10). Pp. viii, 543. $3.50. 1953. Applied Mathematics Series, 27 (National Bureau of Standards, Washington)

The greater part of this useful volume is devoted to a revised edition, with the major errors corrected, of James Dodson's unique *Anti-Logarithmic Canon*, published in 1742. Table I (pp. 2–501) gives 11-figure (10-decimal) values of 10^x for $x = 0(0{\cdot}00001)1$, without differences. Table II (pp. 504–543)

is an original radix table giving 16-figure (15-decimal) antilogarithms of $n \cdot 10^{-p}$, where $n = 1(1)999$ and $p = 3(3)15$.

The rearrangement of the main table so that successive antilogarithms follow one another in a vertical column, instead of in a horizontal line as in Dodson, is a great improvement. On the other hand, the reviewer would have preferred the antilogarithms to be given without the decimal point (as in Shortrede, Filipowski and Deprez), either in full or with the three leading figures outstanding; the dropping of just the first figure except in every tenth line he finds rather disconcerting. The introduction also might have been clearer if it had alluded to the number of *decimals* in logarithms and the number of *figures* in antilogarithms. Regret at the discarding of first differences must be qualified by the admission that the consequent saving of space is considerable. The second differences, which run from 5 to 53 approximately, are not given, although there is ample room in the page headings to have incorporated a mean second difference for each page, or some similar information. The reviewer would have preferred to use this, along with an ordinary small table of the second-difference correction, rather than the graphical schedule on p. vi.

Personal preferences must not be overstressed in face of such an extensive and useful table. Dodson's table was, as he intended, a great work, fit to be mentioned in the same breath as the classical tables of Briggs, de Decker and Vlacq published between 1624 and 1628. Its reissue is an event of importance. It should, however, be clearly understood that Table I has not been entirely recomputed. It results from differencing Dodson's table and correcting errors so revealed. It is stated that about 250 major errors were discovered and corrected; these are not listed. The errors in Dodson have hitherto been only partially known; the differencing is a piece of work which needed to be done. Naturally, it does not reveal rounding errors in the last figure, on which the introduction to the present volume might reasonably have been expected to vouchsafe some information, however brief.

It is only necessary to compare the 16-figure antilogarithms of 0(0·001)1 in Table II with the corresponding 11-figure values in Table I to gain some idea of the errors in the latter. As it may be assumed that all but last-figure errors have been eliminated through differencing, the reviewer concentrated on the last figure of Table I and the last six of Table II; he also compared the last figures of Table I with those of Dodson in several hundred cases, without finding any discrepancy. At 3-decimal arguments in Table I there appear to be no last-figure errors more serious than a number due to faulty rounding. The extremes are at 0·721, where 39 07062 in Table II is rounded up to 40 in Table I, and at 0·419, where 3 84442 is rounded down to 3. Thus the thousand antilogarithms mentioned provide no evidence of error greater than 0·93 units of the last figure. Both the foregoing extremes are exceptional; the fractions of a unit of the eleventh figure lie well outside the range of the other misrounded fractions of a final unit (as we must call them, whether we use the ordinary or Dodson's rule of rounding). Omitting them, the extremes are at 0·859, where 0 21703 is rounded up to 1, and at 0·518, where 7 74578 is rounded down to 7. Had it not been for a systematic error in the antilogarithms for values of x between 0·85 and 0·90 (where one of Dodson's key values may possibly have been too great by three or four units of the twelfth figure, so that about a dozen abnormal misroundings result), the first extreme would even have been that 2 42943 is rounded up to 3 at 0·096.

It seems from Dodson's text that he aimed at raising the last figure when the following (unprinted) figure was 6 or more (not 5 or more, as is most usual). Except for the effects of his unfortunate systematic error and the two exceptional errors, the last figures at 3-decimal arguments in Table I are such

as would result from rounding, according to Dodson's rule, working values never erroneous by more than two units of the twelfth figure. Of the thousand antilogarithms considered, about 16 per cent are wrongly rounded on the usual convention, and about 7 per cent on Dodson's convention ; the rounding errors are minimized, at about 6 per cent, if the rule is supposed to be that the last figure is raised when the following three figures are 633 or more. Thus the last figures have, on the usual convention, a small negative bias of about 0·1 final units.

Published material also exists which gives further figures for a large number of 4-decimal and 5-decimal arguments. It might have been better to give accurately-rounded endings in Table I when they were known, and Dodson's endings when further information was lacking.

Rounding errors are not of great importance in computations with as many as eleven figures. On the other hand, most table-users prefer to know something about the accuracy of the last figure and to be aware of anything unusual about the rounding convention. They might at least have had their attention drawn to the rule which Dodson followed as well as he could. Perhaps the N.B.S. workers, with the powerful equipment at their disposal, may yet produce statistics resulting from a comparison of Table I with such antilogarithms, at the arguments of Table I, as are known to more than eleven figures. It would be interesting to know whether any antilogarithm in Table I differs from the true value by more than one final unit ; as stated above, the reviewer has spotted no discrepancy greater than 0·93 final units.

A. FLETCHER.

Tables of Coefficients for the Numerical Calculation of Laplace Transforms. By H. E. SALZER. Pp. ii, 36. 25 cents. 1953. Applied Mathematics Series, 30 (National Bureau of Standards, Washington)

This publication contains two tables designed to facilitate the numerical calculation of the Laplace transform of $f(t)$, here defined as

$$\int_0^\infty e^{-pt} f(t)\,dt.$$

Table I relates to the case in which the numerical values of $f(t)$ are given at $t=0, 1, \ldots, n-1$, and the function $f(t)$ may be adequately approximated by a polynomial of degree $n-1$. For $n=2(1)11$ the algebraic formulae for the n-point Lagrange interpolation coefficients and their Laplace transforms are set out in full in two useful schedules, and the transforms are tabulated numerically to about eight or nine significant figures for values of p up to $n-1$ at equal intervals (0·1 for the lower values of n).

This gives a rapid process for the calculation of a transform. If $f(t)$ is actually a polynomial (of degree not more than 10) in the range $(0, \infty)$, the process is exact in principle, and the accuracy of the result is limited only by the extent and precision of the table. If $f(t)$ is not a polynomial, it is evidently desirable to consider the adequacy of the approximation in the ranges $(0, n-1)$ and $(n-1, \infty)$; in the latter, one hopes that even spectacular misrepresentation of $f(t)$ will be glossed over adequately (for numerical purposes) by a sufficiently small exponential factor. This reviewer did not succeed in convincing himself that the treatment given in the introduction is correct. Numerical examples, however, show that Table I is capable of giving useful results when it is employed with discretion.

Table II is merely an 8-figure table of $n!/p^{n+1}$, the transform of t^n, for $n=0(1)10$ and $p=0{\cdot}1(0{\cdot}1)10$. It is intended for use when $f(t)$ is given as a polynomial with numerical coefficients.

The author remarks that a different quadrature process, using the zeros and weight factors of the Laguerre polynomials, gives greater accuracy, but requires the numerical evaluation of $f(t)$ at abscissae which are unevenly spaced and moreover different for each p. The reviewer would certainly prefer to use the present convenient tables when they yield a sufficiently accurate result. A. FLETCHER.

Table of Natural Logarithms for Arguments Between Zero and Five to Sixteen Decimal Places. Pp. x, 501. \$3.25. 1953. Applied Mathematics Series, 31 (National Bureau of Standards, Washington)

In 1941 the Mathematical Tables Project issued four volumes of natural logarithms, two giving logarithms of the integers up to 100,000 and two giving logarithms of 0(0·0001)10. As the values of the logarithms in the two sets differ by a constant, namely 4 $\log_e 10$, it has been decided not to reissue the first two volumes. It may be added that other available tables of natural logarithms of integers suffice for most purposes. The present volume, like the original MT10, gives 16-decimal logarithms of 0(0·0001)5, without differences. The introductory matter, however, has been altered, largely by omission. The logarithms of 5(0·0001)10 are also to be reissued in the Applied Mathematics Series. Many who did not obtain the original publications will rejoice to find that they have another chance of acquiring these outstanding fundamental tables. A. FLETCHER.

Mathemagic. By ROYAL VALE HEATH. Pp. 126. Paper \$1. 1953. (Dover Publications, New York)

A collection of " Think of a number " tricks, odd pieces of arithmetical information, magic squares and " Believe it or not " problems dished up with an excessive amount of facetiousness and no algebra. J. C. W. D.

Vector and Tensor Analysis. By G. E. HAY. Pp. viii, 193. \$1.50 ; cloth \$2.75. 1954. (Dover Publications, New York)

This is an introduction assuming no previous knowledge of either discipline. The treatment is rather uneven, in some places elementary manipulation is given in full, in others more elaborate intermediate steps are omitted in a way which would be quite acceptable were not the final result incorrect.

The vector section occupies five chapters and 156 pages ; Elementary Operations, Applications to Geometry, to Mechanics, Partial Differentiation and Integration. The vector methods and results are sound and sometimes elegant but the applications are less good and sometimes erroneous.

The last chapter of 37 pages is devoted to Tensor Analysis as far as covariant differentiation and the curvature tensor. Tensors are linked up with the vectors of the earlier part through Cartesian tensors. J. C. W. D.

The Mathematical Solution of Engineering Problems. By M. G. SALVADORI and K. S. MILLER. Pp. x, 245. 34s. 1954. (Columbia University Press ; London, Geoffrey Cumberlege)

This book is best described by a summary of the contents of its six chapters. I. *A review of some basic mathematical concepts.* The elements of complex numbers, limits, differentiation and integration. II. *Plane analytic geometry.* The simplest equations of the ellipse, hyperbola and parabola. Gradients, curvature in cartesian coordinates, maxima and minima. III. *Numerical solution of algebraic and transcendental equations.* Equations in one unknown. Location of roots, successive approximation. Newton's method. IV. *Numerical solution of simultaneous linear algebraic equations.* Solution by means

of determinants. Pivotal condensation of a determinant. Gauss' method of elimination. Iterative methods. V. *Elementary functions and power series.* Partial fractions. Inverse trigonometric functions. Exponential and logarithmic functions. Binomial series; Maclaurin's and Taylor's series (a formula for the remainder is given). Convergence; comparison test, ratio test, alternating series. Evaluation of sin z, cos z, sinh z, cosh z, e^z, log z (z complex). VI. *Fourier series and harmonic analysis.*

The sequence is not strictly progressive. For instance, short tables of derivatives and integrals in Chapter I involve hyperbolic functions and inverse trigonometric functions. It is not made clear that the formulae given apply to principal values, which are not defined until Chapter V.

The method of presentation is that " a simple physical problem first motivates the introduction of each mathematical technique and of the corresponding theory ". Some of these introductory problems have the character of " adventures from real life ", as for instance that which allows us to deduce from a pilot's last messages that he has crashed at some point P, where $AP+PB=$ constant. The most artificial, used to introduce de l'Hopital's rule, asserts that a company's statistician finds that the profits of two stores on day t are given by the equations

$$y_1=157(t-15)/15 \text{ and } y_2=100 \sin \{\pi(t-15)/60\}.$$

The owners wish to know the ratio of the profits on each day and are baffled on the fifteenth day. One feels that they would be better occupied in seeking means of rectifying the sine-wave. This section includes a dangerously abbreviated table of indeterminate forms. There are several statements such as " If $\phi/f=\infty, f-\phi=-\infty$ ".

There is quite a good collection of examples for the reader at the end of each chapter, starting with straight-forward drill in technique and passing to problems very suitable for engineering students. Some of these might well have been solved as illustrative examples in the text, in preference, or in addition, to the easier ones chosen. Answers are given to about half of the questions.

The book can hardly be recommended to the English technical student taking any of the recognised courses, since there are many (and cheaper) textbooks better suited to his purpose. It might, however, appeal to an engineer or physicist wishing to revise and extend his equipment in mathematics.

C. G. P.

An Introduction to the Theory of Numbers. By G. H. Hardy and E. M. Wright. 3rd edition. Pp. xvi, 419. 42s. 1955. (Geoffrey Cumberlege, Oxford University Press).

" Hardy and Wright " is an established classic, and general recommendation of this third edition would be superfluous. We need only note that the most substantial change is the extension of the third of the chapters on primes to include Selberg's proof of the prime number theorem. This proof is " elementary " in the technical sense, since it makes no appeal to the theory of the Riemann zeta-function. Following the first proofs, by Hadamard and de la Vallée Poussin, steady sapping had reduced the appeal to function theory to what was believed to be the irreducible minimum, namely that $\zeta(s)$ has no zeros on the line $\sigma=1$, so that the Selberg-Erdos proof was something of a sensation. But it must be remembered that proofs which are " elementary " in this technical sense are often more difficult to find and to grasp than the non-elementary arguments.

Professor Wright is surely too modest when in his preface he attributes to himself only the faults of the book; if this were true, the determination of his own contributions would be extremely difficult.

T. A. A. B.

Methods of Mathematical Physics. I. By R. Courant and D. Hilbert. First English edition. Pp. xv, 561. \$9.50. 1953. (Interscience Publishers, New York and London.)

The mathematical event of 1924 was, possibly, the publication of the first edition of Volume I of " Courant-Hilbert ". Some mathematical physicists may have opened their eyes widely at the emphasis on what we now call linear analysis and on the connection of eigen values with variational principles; but the prescience of the authors was fully justified by succeeding developments, so that now " Courant-Hilbert " is not a survey preliminary to exploitation but a map of an organised and settled domain. Its value is thus changed in type but not diminished, and an English translation is very welcome to those who have not the gift of tongues. Courant himself has prepared the translation and has made some additions and improvements, but the book is substantially the equivalent of the second (1931) German edition.

The printing is good and clear, being more generously spaced than in the Springer volumes, but Interscience Publishers might well improve their standard, good though it is, by avoiding unsightly " rules " and by a more consistent use of small fractions.

My one regret is that this version has necessarily delayed Courant in the preparation (in collaboration with K. O. Friedrichs) of the completely revised and modernised " Courant-Hilbert " which he has promised us. May we hope that when the translation of Volume II is off his hands, he will press on rapidly to the fulfilment of that promise. T. A. A. B.

A Concise History of Mathematics. By D. J. Struik. Pp. xix, 299. 14s. 1954. (Bell)

This volume, a British edition of an American work reviewed in the *Gazette*, December 1953, is a marvel of compression. The author has concentrated on main lines of development, stopping at the end of the nineteenth century, in order to compress what he has to say into 300 pages; he also assumes that the reader is moderately familiar with the elements of mathematics. Personal details of biography are suppressed, but something of the historical background is supplied. There are 47 plates, mainly portraits of great mathematicians.

The teacher who wisely obtains this book for the school library should be prepared to answer questions which the young reader may raise. What does the author mean by saying (p. 45) that the problem of the quadrature of the circle is " to find the square of an area equal to that of a given square ", or (p. 176) that Euler " simply did not always use some of our present tests of convergence or divergence as a criterium for the validity of his series "? But errors of fact are rare, though I think that Cardan is given less than his due; and the statement about De Morgan in the footnote on p. 182 is unjustifiable.

T. A. A. B.

A Treatise on Conic Sections. By G. Salmon. 6th edition, rep. Pp. xv, 399. Paper covers, \$1.94; cloth \$3.25. 1954. (Chelsea Publishing Company, New York)

In 1920 a boy about to go up to Cambridge was told by a don to read Salmon's *Conics*, not for the sake of the topic but because " every gentleman ought to have read Salmon "; yet even then the *Conics* was over 70 years old. Worse advice could be given today; his books " excel in clarity and charm " and " even now have hardly been surpassed ", says Struik in his *Concise History*, while that cynical iconoclast, E. T. Bell, allows Salmon to be " a fine geometer and algebraist " even though he did desert mathematics for theology. The

Chelsea classics form a goodly company and Salmon takes his place therein as of right; perhaps many of us would like to see the *Higher plane curves* added to the list of reprints. T. A. A. B.

Theory of Functions of a Complex Variable. I. By C. Carathéodory. Translated by F. Steinhardt. Pp. xii, 301. \$5. 1954. (Chelsea Co., New York)

We must thank the Chelsea Company for their enterprise in producing a translation of this, the last, and if not the most important, then perhaps the most charming, of Carathéodory's books.

After a formal introduction of the complex number, a good deal of space is given to the bilinear (Möbius) transformation and the geometry of inversion to which it is so closely related; this leads to euclidean, spherical and non-euclidean geometry and trigonometry on what is perhaps a side-track, but a pleasant one, before we come back on to the main road at a utilitarian account of matters in point set theory and topology. This concise account is a necessary preliminary to the contour integral, Cauchy's theorem, the maximum modulus principle, Poisson's integral and harmonic functions. By-passing most of the calculus of residues, there is a long and valuable chapter on analytic functions defined by limiting processes, including power series, the Taylor and Laurent expansions, Mittag-Leffler's theorem and kindred matters. Here the use of the concept of continuous convergence, Ostrowski's limiting oscillation, and Montel's normal families has effected a considerable simplification and economy. The final section discusses the elementary functions: the exponential, logarithm and Gamma functions.

The translation is often stiff but seldom obscure. The printing is clear, though hardly so polished as in the original Birkhäuser edition. The book is a close rival to that of Ahlfors for the honour of being the best modern introduction to the theory of the complex variable, and we hope that the Chelsea Company will soon offer us a translation of Volume II. T. A. A. B.

FILM STRIPS

The Use of Graphs. By G. H. Grattan-Guinness and H. V. Lowry. Part 1 Mathematical Pictures; Part 2, Graphs of Relationships; each part 24 frames, 12s. 6d. (*Picture Post*)

Many teachers will have felt the need for a collection of ready-made graphs to illustrate their work in the development of this topic. In these two filmstrips they will find a useful collection of graphs and diagrams, set out in a logical sequence, to provide a background for the pupil's own work. Part I begins with the pictorial form, the school time table being illustrated by bar, sector, area and isotype. The rest of this strip is devoted to the axes and co-ordinates, including some interesting frames on the National Grid and the effects of different scales on the two axes. Part 2 deals with the straight line, parabola, rectangular hyperbola and some trigonometrical functions. The work is skilfully planned, and there are many useful ideas to be found in the teaching notes. The usual criticism, true of many earlier filmstrips, of slightness of treatment, cannot be made here, for the mathematical content is high enough for most classes. The drawings and photography are of the quality one expects from this publication. I. R. V.

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WILLIAM VALLANCE DOUGLAS HODGE, Sc.D., F.R.S.

President, January 1954—April 1955